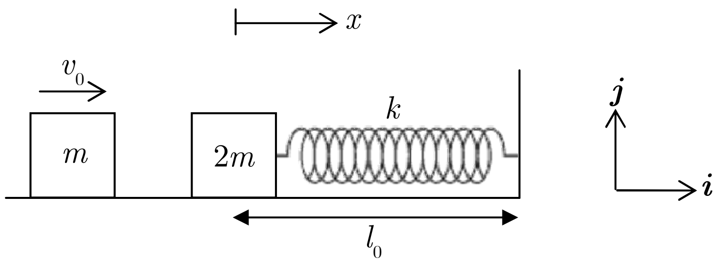
Physics 8.01 T – Section L05

Solutions to Quiz 4

IMPORTANT NOTE: This quiz was found by many to be more difficult than previous quizzes, and it certainly is richer conceptually. I have therefore made the solutions much more detailed, and I've added a number of supplementary comments you might find useful in understanding the material. Reading these comments, however, is *completely optional* and their content was certainly not required for credit in the quiz. I have identified all optional material with a *grey* background.

We first define our coordinate system



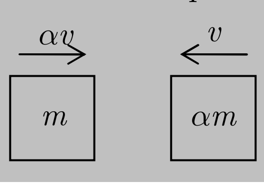
Part A

The collision is inelastic, because the blocks stick to each other after the collision and kinetic energy is therefore lost.

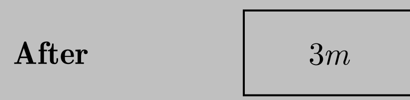
Note: Many students often wonder *why* it is so obvious that if the two particles stick together during the collision, kinetic energy is always lost. A rather cute argument to show this is as follows:

- We can consider the collision in *any* frame – therefore, let's consider it in a frame in which the *total momentum* is 0 [this is known as the **centre-of-mass frame**]. This can easily be done by moving along with the particles in such a way that the momentum of each particle looks equal, but opposite in direction to that of the other particle. For example:

Before



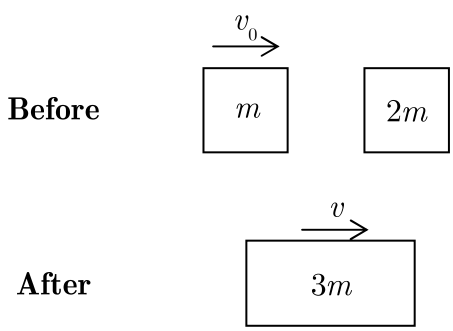
- After the collision, only one particle is left. However, the total momentum before the collision was 0, so the total momentum after the collision must also be 0. Therefore, the final particle must be stationary!



- But clearly, before the collision, there *was* some kinetic energy. After the collision, there was *no* kinetic energy. Therefore, kinetic energy is definitely lost.
- You now understand why such a collision is called *totally inelastic*. Because in the centre-of-mass frame, all the kinetic energy is lost.

Part B

There are no external forces, so we need to conserve momentum in this collision.



In the ***i*** direction, therefore, we get:

$$\text{Momentum after} - \text{Momentum before} = 0$$
$$3mv - mv_0 = 0$$
$$\boxed{v = \frac{1}{3}v_0}$$

Part C

The angular frequency of the oscillation is $\omega = \sqrt{k / 3m}$, and the period is therefore $T = 2\pi / \omega$. Originally, the particles are at the *equilibrium* position, exactly at the centre of the motion. They will therefore take a time *T/4* to get to the "fully compressed" point. As such

$$t = \frac{T}{4} = \frac{2\pi}{4\omega}$$
$$\boxed{t = \frac{\pi}{2}\sqrt{\frac{3m}{k}}}$$

To find the shortest length of the spring, we use conservation of energy. If the minimum length of the spring is L, then its extension is $l_0 - L$, and so

$$\text{Energy just after collision} = \text{Energy when maximally compressed}$$

$$\frac{1}{2}(3m)v^2 = \frac{1}{2}k(l_0 - L)^2$$

$$3m\left(\frac{1}{3}v_0\right)^2 = k(l_0 - L)^2$$

$$m\frac{1}{3}v_0^2 = k(l_0 - L)^2$$

$$\sqrt{\frac{mv_0^2}{3k}} = l_0 - L$$

$$\boxed{L = l_0 - v_0\sqrt{\frac{m}{3k}}}$$

Many people gave me the *extension* in the spring rather than its length (ie: $v_0\sqrt{m/3k}$ instead of $l_0 - v_0\sqrt{m/3k}$. This was a small misreading of the question, and it wouldn't have cost you too many points.

Many people tried to do this problem using the equations of simple-harmonic motion directly, and inevitably failed. This is *definitely* not the most efficient way to do this problem, but I'll do it here just to show you how it *should* have been done. The general solution of the SHM equation is

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x(t) = A\cos\left(t\sqrt{\frac{k}{3m}}\right) + B\sin\left(t\sqrt{\frac{k}{3m}}\right)$$

We take our origin at the equilibrium position, and our boundary conditions are therefore [see important footnote[1]]

[1] *The main mistake people made was not to take the origin at the equilibrium position. They then ended up assuming that at t = 0, x = l_0. This is a terrible idea for two reasons*

- *It makes the algebra excruciatingly complicated.*
- *It's just plain wrong, because you have assumed that the spring force is given by F = –kx. However, if you don't take x from the equilibrium position, this formula is no longer true, because "x" in the formula refers to the extension of the spring from equilibrium.*

The moral of the story is that in SHM, you should [almost] ***always*** *take your origin at equilibrium.*

- At $t = 0$, $x = 0$.
- At $t = 0$, $v = v_0/3$ (found above, in part B)

These two boundary conditions give us [using the standard methods derived in the lecture]

$$A = 0 \qquad\qquad B = \frac{v_0}{3}\sqrt{\frac{3m}{k}} = v_0\sqrt{\frac{m}{3k}}$$

And so

$$x(t) = v_0\sqrt{\frac{m}{3k}}\sin\left(t\sqrt{\frac{k}{3m}}\right)$$

At the point of maximum compression, $v(t) = 0$, because the block has stopped[2], and so

$$v(t) = x'(t) = \frac{v_0}{3}\cos\left(t\sqrt{\frac{k}{3m}}\right) = 0$$

This implies that

$$t\sqrt{\frac{k}{3m}} = \frac{\pi}{2}$$
$$t = \frac{\pi}{2}\sqrt{\frac{3m}{k}}$$

As above, and at that value of t

$$x(t) = v_0\sqrt{\frac{m}{3k}}\sin\left(\frac{\pi}{2}\sqrt{\frac{3m}{k}}\times\sqrt{\frac{k}{3m}}\right) = v_0\sqrt{\frac{m}{3k}}\sin\left(\frac{\pi}{2}\right) = v_0\sqrt{\frac{m}{3k}}$$

Again, in agreement with what we found above.

But in any case, I hope I've convinced you that the first method was rather easier!

Part D

We simply use the fact that

Impulse = Change in momentum

Originally, the blocks are travelling at speed v in the positive $\boldsymbol{i}$ direction. By the time they've stopped, they're travelling at speed 0. Therefore

[2] You could also note that at the point of maximal compression, x must be at a maximum, and so the sin term must be equal to 1. All the results below follow more quickly.

$$\begin{aligned}\text{Impulse} &= \text{Final momentum} - \text{Original momentum}\\ &= 0 - \left(3mv\boldsymbol{i}\right)\\ &= -3m\frac{1}{3}v_0\boldsymbol{i}\\ &= -mv_0\boldsymbol{i}\end{aligned}$$

So:

$$\boxed{\text{Impulse} = -mv_0\boldsymbol{i}}$$

So the magnitude of the impulse is mv_0 and its acts in the negative $\boldsymbol{i}$ direction.

One of the most common mistakes I saw in this part of the problem was to use the constant-acceleration equations or a variation thereof. There is *no way* that this is valid here, because the acceleration is clearly changing as the particle moves.

Another (*much, much* more complicated) way to solve this problem, which I often saw people attempt, is to use the expression $\boldsymbol{I} = \int \boldsymbol{F}\,\mathrm{d}t$ directly. I'll do it here just to show you how it's done. You'll have got full points if this is what you did in the quiz, but it's really *not* the best way to do the problem! First, since the integral is dt, we need an expression for the force in terms of time

$$\boldsymbol{F} = -kx\boldsymbol{i}$$

$$\boldsymbol{F} = -k\left(A\cos\sqrt{\frac{k}{3m}}t + B\sin\sqrt{\frac{k}{3m}}t\right)\boldsymbol{i}$$

At $t = 0$, $F = 0$, because the blocks are at equilibrium, and so $A = 0$. The velocity at $t = 0$ is v, and so

$$\begin{aligned}v = \frac{v_0}{3} &= \cancel{-A\sqrt{\frac{k}{3m}}\sin\left(0\times\sqrt{\frac{k}{3m}}\right)} + B\sqrt{\frac{k}{3m}}\cos\left(0\times\sqrt{\frac{k}{3m}}\right)\\ &= B\sqrt{\frac{k}{3m}}\\ \Rightarrow B &= \frac{v_0}{3}\sqrt{\frac{3m}{k}}\end{aligned}$$

As such

$$\boldsymbol{F} = -\left(\frac{kv_0}{3}\sqrt{\frac{3m}{k}}\sin\left(t\sqrt{\frac{k}{3m}}\right)\right)\boldsymbol{i}$$

We found above that the time it takes to compress completely is

$$t = \frac{\pi}{2}\sqrt{\frac{3m}{k}}$$

And so the impulse is

$$\begin{aligned}
\boldsymbol{I} &= \int_0^{\frac{\pi}{2}\sqrt{\frac{3m}{k}}} \boldsymbol{F}\,\mathrm{d}t \\
&= -\frac{kv_0}{3}\sqrt{\frac{3m}{k}}\int_0^{\frac{\pi}{2}\sqrt{\frac{3m}{k}}} \sin\left(t\sqrt{\frac{k}{3m}}\right)\boldsymbol{i}\,\mathrm{d}t \\
&= \frac{kv_0}{3}\sqrt{\frac{3m}{k}}\left[\sqrt{\frac{3m}{k}}\cos\left(t\sqrt{\frac{k}{3m}}\right)\right]_0^{\frac{\pi}{2}\sqrt{\frac{3m}{k}}}\boldsymbol{i} \\
&= mv_0\left[\cos\left(\frac{\pi}{2}\sqrt{\frac{3m}{k}}\sqrt{\frac{k}{3m}}\right) - \cos\left(0\times\sqrt{\frac{k}{3m}}\right)\right]\boldsymbol{i} \\
&= mv_0\left[0-1\right]\boldsymbol{i} \\
&= -mv_0\boldsymbol{i}
\end{aligned}$$

Exactly the same result as above. But I hope I've convinced you the first method was better!

Once again, many people failed in this approach because they tried to take their origin at a position other than the equilibrium position.

Part E

The impulse exerted by the spring will be *less* than it would be without friction. The best way to argue this answer is as follows:

- The change in momentum of the blocks is still the same, so the total impulse on the blocks is still the same.
- However, there is now an additional impulse provided by friction as the block moves.
- Therefore, the spring won't have to provide as much impulse to get the block to stop.
- Therefore, the spring provides less impulse than without friction.

The most common error I spotted here was miss-reading the question. We did not ask you how the *total* impulse would change, we asked you how the impulse *exerted by the spring* would change

An argument many of you gave was as follows:

- Because of friction, the block takes a shorter time to get to the compressed position.
- Therefore, in $I = \int F \, \mathrm{d}t$, the time interval is shorter, and so the impulse is less.

There's a slight gap in this argument – it's possible that even though the time is shorter, F might be much bigger. To make the argument complete, you really should have mentioned the fact that the *force* from the spring doesn't change as a result of the friction, and only depends on x. However, in grading the quiz, I assumed that you all knew that, and I gave you the points anyway if you presented this argument.

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