# B6015 – Decision Models
# Review Session 3

The primary aims of this review session are:

- To get practice with logical constraints, using integer linear programming, as introduced in class.
- To become familiar with transportation problems.
- To understand how complex constraints can be factored into transportation problems.

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You are the primary decision maker for Omazan, one of the premier online book retailers in the United States. Though your company ships to any location in the United States, your operations are mostly centered around four cities - New York (NY), Los Angeles (LA), Chicago (CH) and Houston (HO); for the purposes of this question, ignore all other cities. The latest Harry Potter book has just come out, and you expect record sales. Table 1 lists the demand for the new book at each city.

| | Demand (thousands) |
|---|---|
| NY | 100 |
| LA | 30 |
| CH | 40 |
| HO | 45 |

Table 1: Demand for books.

You have suppliers of the book in every city. However, due to regional differences, these suppliers charge different amounts for each book. Table 2 lists the price charged by each supplier.

| | Price per book ($) |
|---|---|
| NY | 9 |
| LA | 4 |
| CH | 6 |
| HO | 5 |

Table 2: Price of books.

The book is so eagerly awaited that the publisher has made it available to retailers before the official release date, to allow them to build up appropriate stock. Omazan decides to take advantage of this opportunity and to ensure each city has adequate stock to meet demand well in advance of the release date. To achieve this aim, Omazan decides to buy some books at each of these four cities and perform whatever shipping is necessary to ensure each city has enough books to meet its demands.

Omazan uses a local shipping company, EedFx, with which it has a long-standing relationship. The prices EedFx charges for shipping books from city to city is given in Table 3 (notice that shipping prices are symmetric - the price of shipping one book from NY to LA is the same as the price of shipping one book from LA to NY). EedFx is large enough that it is able to ship an effectively infinite number of books from city to city.

| | NY | LA | CH | HO |
|---|---|---|---|---|
| NY | – | 3.5 | 2.5 | 2.5 |
| LA | 3.5 | – | 2.5 | 2.5 |
| CH | 2.5 | 2.5 | – | 1.5 |
| HO | 2.5 | 2.5 | 1.5 | – |

Table 3: Shipping costs ($ per book).

You are, however, anticipating that the NY $\leftrightarrow$ LA shipping route will be particularly popular, and in addition to EedFx's services, you are considering two other companies as well:

- Company A, which can transport up to 40,000 (total) books on the NY $\leftrightarrow$ LA route at a cost of $ 2 per book transported. Due to administrative costs, however, you estimate that starting business with this new company will necessitate a $ 10,000 initial investment.
- Company B, which can transport up to 20,000 (total) books on the NY $\leftrightarrow$ LA route at a cost of $ 2 per book transported, with only a $ 5,000 initial investment.

You can choose to start business with either company, or neither, but not both.

## Part A

As the decision maker for Omazan, decide

- How many books to buy from each supplier,
- what shipping schedule to use to ensure that demand is completely met, and
- whether to enter into a contract with one of the new companies for shipment of books between NY and LA, and if so which one,

while minimizing Omazan's costs and ensuring demand is completely met. You may assume that any number of books, even fractional, can be purchased and transported.

### Solution

As usual, let us follow our three steps. We first need to decide what the decision variables will be for this problem. As per usual, all amounts will be in thousands of dollars, and decision variables will be in thousands of books. The last part of the question gives us a hint by listing "what we need to decide". Our decision variables will be:

- $S_{NY}$, $S_{LA}$, etc... – these variables will hold the total number of thousands of books we will be buying from each of the four locations. These variables will be constrained to be positive – we can only buy books, not sell them.
- $T_{NY \to LA}$, $T_{NY \to CH}$, etc... – these variables will hold the total number of thousands of books we intend to ship between cities. For example, variable $T_{NY \to LA}$ will hold the number of thousands of books we will be shipping from NY to LA. There will be 12 of these variables in total. These variables will also have to be positive, because we cannot ship book backwards.[1]
- $A$ and $B$ will be a binary variables, which will hold the following values:

  $$A = \begin{cases} 1 & \text{if we enter into a contract with company A} \\ 0 & \text{otherwise} \end{cases}$$

  $B$ will be similarly defined for company B. $a_{NY \to LA}$ will be the number of units shipped from NY to LA using company A, and $a_{LA \to NY}$ vice versa. Similar variables will be defined for company B.

[1] Note that since shipping costs are symmetric, it is also possible to only define 6 variables $T_{NY \leftrightarrow LA}$, $T_{LA \leftrightarrow CH}$, etc..., which define the total flow between cities in *either* direction. We would then, however, have to allow these variables to take negative values to reflect shipping in the "other direction". This is analogous to the choice we had in review session 1 when considering a situation in which funds could be shorted. We would either use one variable and allow it to go negative, or two variables. In both cases, using the two-variable formulation makes it simpler to express complicated constraints.

We now need to consider our objective function, which will be as follows:

$$\begin{aligned} \text{Objective} &= \text{Price of purchase} + \text{Price of EedFx transport} \\ &\quad +\text{Price of new contract} \\ &\quad +\text{Price of new company transport} \end{aligned}$$

How do we express this objective function in terms of our decision variables?

- The "fixed price of new contract" is simply given by $10A+5B$. Indeed, if we enter into contract A, we will have $A = 1$ and $B = 0$ and this will be equal to 10, and similarly if contract B is entered into. If no contracts are entered into, both $A$ and

$B$ will be equal to 0, and this part of the objective function will be equal to 0.

- The other three terms should be obvious – we simply multiply the cost of each shipping route/book purchase by the decision variable corresponding to the number of books shipped along that route/bought in that place. See below for the full formulation.

Now, for our constraints:

- We require all demand at each city to be met – in other words, the total amount of books bought in a given city minus those shipped out should be equal to the demand at that city.

  These constraints are slightly fiddly to handle correctly. For all cities other than NY and LA, they are simple enough. For example, for CH

$$\overbrace{S_{CH}}^{\text{Bought in CH}} - \overbrace{(T_{CH\to NY} + T_{CH\to LA} + T_{CH\to HO})}^{\text{Shipped out of CH}} + \overbrace{(T_{NY\to CH} + T_{LA\to CH} + T_{HO\to CH})}^{\text{Shipping into CH}} \geq \overbrace{40}^{\text{Demand at CH}}$$

  For NY and LA, the constraints are slightly more complicated, because we must also include shipments form the new companies. For example, for NY

$$\overbrace{S_{NY}}^{\text{Bought in NY}} + \overbrace{(T_{NY\to LA} + T_{NY\to CH} + T_{NY\to HO} + a_{NY\to LA} + b_{NY\to LA})}^{\text{Shipped out of NY}} + \overbrace{(T_{CH\to NY} + T_{LA\to NY} + T_{HO\to NY} + a_{LA\to NY} + b_{LA\to NY})}^{\text{Shipped into NY}} \geq \overbrace{100}^{\text{Demand at NY}}$$

  These constraints look complicated, but they are in fact extremely easy to implement in Excel, provided you set up your decision variables correctly.

- We require $a_{NY\to LA} + a_{LA\to NY}$ (the total amount shipped by company A) to be less than or equal to 40,000, or less than or equal to 0 if we do not enter into a new contract with company A.

  To implement this constraint, we used the "trick" developed in class, and write

$$a_{NY\to LA} + a_{LA\to NY} \leq 40A$$

  If we do not enter into the new shipping contract with company A, then, $A = 0$ and so this constraint effectively states both variables must be 0 (since they must be non-negative). If we do enter into the contract, $A = 1$ and the constraint works as expected.

A similar constraint can be implemented for company B.

- Finally, we need to somehow encode the fact that it is not possible to enter into new contracts with both companies – only with one or the other or neither. This is done by ensuring that the sum of the two binary decision variables $A$ and $B$ be less than or equal to 1.

$$A + B \leq 1$$

This allows neither contract to be entered into (in which case $A + B = 0$) or one of the contracts ($A + B = 1$), but not both.

Combining the above, we find that our full algebraic model is

$$\begin{aligned}
\min \quad & 9S_{NY} + 4S_{LA} + 6S_{CH} + 5S_{HO} \\
& \quad +3.5\left(T_{NY\to LA} + T_{LA\to NY}\right) + 2.5\left(T_{NY\to CH} + T_{CH\to NY}\right) \\
& \quad +2.5\left(T_{NY\to HO} + T_{HO\to NY}\right) + 2.5\left(T_{LA\to CH} + T_{CH\to LA}\right) \\
& \quad +2.5\left(T_{LA\to HO} + T_{HO\to LA}\right) + 1.5\left(T_{CH\to HO} + T_{HO\to CH}\right) \\
& \quad +10A + 2\left(a_{NY\to LA} + a_{LA\to NY}\right) \\
& \quad +5B + 2\left(b_{NY\to LA} + b_{LA\to NY}\right) \\
\text{s.t.} \quad & S_{NY} - \left(T_{NY\to LA} + T_{NY\to CH} + T_{NY\to HO} + a_{NY\to LA} + b_{NY\to LA}\right) \\
& \quad + \left(T_{CH\to NY} + T_{LA\to NY} + T_{HO\to NY} + a_{LA\to NY} + b_{LA\to NY}\right) \geq 100 \\
& S_{LA} - \left(T_{LA\to NY} + T_{LA\to CH} + T_{LA\to HO} + a_{LA\to NY} + b_{LA\to NY}\right) \\
& \quad + \left(T_{CH\to LA} + T_{NY\to LA} + T_{HO\to LA} + a_{NY\to LA} + b_{NY\to LA}\right) \geq 30 \\
& S_{CH} - \left(T_{CH\to NY} + T_{CH\to LA} + T_{CH\to HO}\right) + \left(T_{NY\to CH} + T_{LA\to CH} + T_{HO\to CH}\right) \geq 40 \\
& S_{HO} - \left(T_{HO\to NY} + T_{HO\to LA} + T_{HO\to CH}\right) + \left(T_{NY\to HO} + T_{LA\to HO} + T_{CH\to HO}\right) \geq 45 \\
& a_{NY\to LA} + a_{LA\to NY} \leq 40A \\
& b_{NY\to LA} + b_{LA\to NY} \leq 20B \\
& A + B \leq 1 \\
& \text{All } S, T, a \text{ and } b \text{ positive.} \\
& A \text{ and } B \text{ binary.}
\end{aligned}$$

See the Excel file for the optimized spreadsheet.

## Part B

Remove all integer/binary constraints from your model above, and replace them with constraints ensuring that each one of these variables is between 0 and 1 (so instead of $X$ binary, add the two constraints $X > 0$ and $X < 1$). Re-solve. The solution should stay the same (extra credit: why?) Generate a sensitivity report, and use it to answer the following questions without resolving the model:

1. By how much would the initial fixed cost of initiating business with company A have to increase for it to no longer become profitable to enter into business with it?

2. Suppose you are now allowed to enter into business with both company A and B, if you so wish. Would you choose to do it? If so, by how much would your costs go down?

3. Assume, now that shipping prices no longer need to be symmetric (ie: the NY → LA price no longer needs to be equal to the LA → NY price). By how much would the EedFx shipping cost from LA to NY have to decrease to make it no longer profitable to use company A's services?

4. Company A offers to increase its shipping limit from 40,000 books to 50,000. How much should you be willing to pay for this?

## Solution

These questions all deal with sensitivity analysis. Let's consider each[2]

1. What part of the problem would change if we were to change this initial fixed cost? It is the objective coefficient of the decision variable A. The allowable increase for this coefficient is 25, and so the fixed cost would have to increase by \$ 25,000 for the solution to change.

2. The constraint that would now change is the "number of contracts" constraint, which would become $A + B \leq 2$ rather than $A + B \leq 1$. In other words, its RHS would increase by 1. This is within the allowable increase of 1 given in the sensitivity report, and the shadow price is -25. Thus, allowing both contracts would decrease total costs by \$ 25,000. Clearly, this decrease implies that we *will* choose to enter into both contracts.

3. The part of the problem that would now change is the objective coefficient of the variable $T_{LA \to NY}$. We would like to know by how much it can decrease before the solution changes.

   Looking at our sensitivity report, we find that the LA → NY price would only have to decrease by 1.25 for the solution to change. This would still leave the price positive, and so it would be a valid decrease.

   This is therefore the answer to our question – if the EedFx price of shipping between LA and NY were to decrease by

[2] Note that in items 1 and 3 below, the only thing we can definitely deduce is that the solution would *change* – not that it would result in stopping business with company A. However, common sense will quickly show that severing relations with company A must be an outcome of the change. Note also that though this kind of reasoning is useful, I've never seen it needed in a decision models exam before, and I don't expect it'll be needed in your exam – all the questions we ask you are likely to be more straightforward. I just wanted to use the review session to stretch the envelope slightly and show what is possible.

any more than \$ 1,250, it would no longer be profitable to use company A's services.

4. The constraint in question here is the company A shipping constraint, which is currently given by

$$a_{NY \to LA} + a_{LA \to NY} \leq 40A$$

Ideally, we would like to be able to change the figure 40 above to 50. Unfortunately, this is *not* a change that we can deal with using a sensitivity report, because it involves changing the coefficient of one of the decision variables, not the constant term on the RHS of the equation.

However, in this particular case, it *is* possible to answer this question using a sensitivity report, and this is because in our particular solution, company A *is* used and so $A = 1$. Increasing the capacity of company A at no extra fixed cost could only *increase* our incentive to use that company, so we know that in any new solution, we would also have $A = 1$. For that reason, we can model our change using the following new constraint

$$a_{NY \to LA} + a_{LA \to NY} \leq 40A + 10$$

Since we know $A$ will be equal to 1, the RHS will be 50 as expected.

In terms of the sensitivity report, this is equivalent to a RHS increase of 10 (from 0), which is within the allowable increase of 60. The shadow price is -1.5, so this change will result in a decrease of 15 (equivalent to \$ 15,000) in total cost.

Thus, the company should be willing to pay up to \$ 15,000 for this capacity increase.

## Part C

Just for practice, what constraints would you have to add to your algebraic model to ensure that

1. On the LA/NY route, at least twice as many books are shipped in total using EedFx as are shipped using either new company?
2. The new company is used only if at least 50,000 books are shipped in total using EedFx along the LA/NY route?

Finally, how would you modify your model if the shipping capacity using company A was limitless?

## Solution

To ensure that

1. on the LA/NY route, at least twice as many books are shipped *in total* using EedFx as are shipped using either new company, we would use the constraint

$$2\left(a_{NY\to LA}+a_{LA\to NY}\right)\leq T_{NY\to LA}+T_{LA\to NY}$$

   and a similar constraint for company B.

2. the new company is used only if at least 50,000 books are shipped in total using EedFx along the LA/NY route, we would use the constraint

$$A\leq\frac{X_{NY\to LA}+X_{LA\to NY}}{50}$$

   and a similar constraint for company B. If the total number of books shipped along that route is indeed greater or equal to 50,000, the RHS will be greater than 1 and the constraint will always be satisfied since $N$ is a binary decision variable. If not, the RHS will be less than 1, forcing $N$ to be 0 (ie: forcing us not to enter into the new contract).

Finally, if the shipping capacity using company A was limitless, we would modify the constraint $a_{NY\to LA}+a_{LA\to NY}\leq 40A$, and simply replace 40 by some other very larger number (for example, in this case, it might make sense to use the sum of all the demands to every city – clearly, we'll never need to ship that many books along any route).