Deriving the Power of a Test

Consider a situation in which we would like to test whether an overdose rate has shrunk from p_{old} to $p_{\text{new}} = p_{\text{old}}/2$ (ie: whether the overdose rate has halved). The question is – how large does our sample need to be to get a certain power for a test of a certain size?

Strictly speaking, we should fit a binomial generalized linear model and derive a test statistics from that model. Given the large number of sample, however, we take a "back of the envelope" approach instead.

Let the original overdose rate be p_{old}, the new rate p_{new}, and let n individuals be sampled in each case (in this case, 15, 000) or which d_{old} and d_{new} die of overdose. The hypotheses we want to test are

$$H_0 : p_{\text{new}} = p_{\text{old}}$$
$$H_1 : p_{\text{new}} = p_{\text{old}} / 2$$

Now, note that

$$d_{\text{old}} \sim \text{Bin}\left(n, p_{\text{old}}\right) \Rightarrow \mathbb{V}\text{ar}\left(d_{\text{old}}\right) = np_{\text{old}}\left(1 - p_{\text{old}}\right) = np_{\text{old}}(1 - p_{\text{old}})$$

Since the rates are very small and n is very large, we approximate

$$d_{\text{old}} \sim N\left(np_{\text{old}}, np_{\text{old}}\overline{p}_{\text{old}}\right)$$

and similarly for d_{new}. This means that

$$d_{\text{old}} - d_{\text{new}} \sim N\left(n\left\{p_{\text{old}} - p_{\text{new}}\right\}, n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})\right)$$

Our test statistic will therefore be

$$Z = \frac{d_{\text{old}} - d_{\text{new}}}{\sqrt{n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})}} \sim \begin{cases} N(0,1) & \text{under } H_0 \\ N(np_{\text{new}},1) & \text{under } H_1 \end{cases}$$

Now, if we want the significance of our test to be α,

$$\mathbb{P}\left(H_0 \text{ rejected} \mid H_0 \text{ true}\right) = \alpha$$
$$\mathbb{P}\left(\frac{d_{\text{old}} - d_{\text{new}}}{\sqrt{n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})}} > C\right) = \alpha$$
$$C = z_\alpha$$

All that said, the power of our test, $1 - \beta$, is given by

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$$\mathbb{P}\left(H_0 \text{ accepted} \mid H_1 \text{ true}\right) = \beta$$

$$\mathbb{P}\left(\frac{d_{\text{old}} - d_{\text{new}}}{\sqrt{n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})}} > C\right) = 1 - \beta$$

$$\mathbb{P}\left(\frac{d_{\text{old}} - d_{\text{new}} - np_{\text{new}}}{\sqrt{n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})}} > C - \frac{np_{\text{new}}}{\sqrt{n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})}}\right) = 1 - \beta$$

$$C - \frac{np_{\text{new}}}{\sqrt{n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})}} = z_{1-\beta}$$

Solving to eliminate C

$$z_{\alpha} - z_{1-\beta} = \frac{np_{\text{new}}}{\sqrt{n(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}})}}$$

$$\frac{\left(z_{\alpha} - z_{1-\beta}\right)^2\left(p_{\text{old}}\overline{p}_{\text{old}} + p_{\text{new}}\overline{p}_{\text{new}}\right)}{p_{\text{new}}^2} = n$$