Actuarial Statistics – 2006 Paper

Question 1

1 Consider the total amount of the claims arising from traffic accidents for which an insurance company receives at least one claim. Let N be the number of claims from one such accident and let the claim sizes $X_1, X_2, \ldots$ be independent identically distributed random variables, independent of N. Let $p_n = \mathbb{P}(N = n)$, so that $p_0 = 0$, and assume that

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \quad n = 2, 3, \ldots,$$

where $a, b \in \mathbb{R}$ are known.

Suppose that the claim sizes are discrete with $f_k = \mathbb{P}(X_1 = k)$, $k = 1, 2, \ldots$, where $\sum_{k=1}^{\infty} f_k = 1$, and assume that the p_n's and the f_k's are known. Let $g_k = \mathbb{P}(X_1 + \ldots + X_N = k)$, $k = 1, 2, \ldots$.

By considering probability generating functions, derive a recursion formula for the g_k's in terms of known quantities.

Write down the recursion if

$$p_n = \frac{e^{-\lambda}\lambda^n}{(1 - e^{-\lambda})n!} \quad n = 1, 2, \ldots$$

We begin by noting that since the claim sizes cannot be 0, $g_0 = \mathbb{P}(N = 0) = p_0 = 0$.

We also note that for the compound distribution to have a value of 1, we must have a *single* claim, with a value of 1. So $g_1 = f_1 p_1$. This will be the basis for our recursion formula.

Now, mutiply the condition in the question by z^n and sum, to get

$$\begin{aligned}
\sum_{n=1}^{\infty} p_n z^n &= \sum_{n=1}^{\infty} z^n \left(a + \frac{b}{n}\right) p_{n-1} \\
\sum_{n=0}^{\infty} p_n z^n - p_0 &= \sum_{n=1}^{\infty} azz^{n-1} p_{n-1} + b\sum_{n=1}^{\infty} \frac{z^n}{n} p_{n-1} \\
G_N(z) - p_0 &= az\sum_{n=0}^{\infty} z^n p_n + b\sum_{n=1}^{\infty} \frac{z^n}{n} p_{n-1} \\
\left(1 - az\right) G_N(z) &= p_0 + b\sum_{n=1}^{\infty} \frac{z^n}{n} p_{n-1}
\end{aligned}$$

Differentiating with respect to z

$$-aG_N(z) + \left(1 - az\right) G_N'(z) = bG_N(z)$$

$$G_N'(z) = \frac{a + b}{1 - az} G_N(z)$$

Now, let

$$G_S(z) = \sum_{n=0}^{\infty} g_n z^n$$

We have $M_S(u) = G_N\left(M_X(u)\right)$, and we also know that $G(z) = M\left(\log z\right)$, so

$$G_S(z) = M_S\left(\log z\right) = G_N\left(M_X(\log z)\right) = G_N\left(G_X(z)\right)$$

Differentiating, we get

$$\begin{aligned} G_S'(z) &= G_N'\left(G_X(z)\right)G_X'(z) \\ &= \frac{a+b}{1-aG_X(z)}G_N\left(G_X(z)\right)G_X'(z) \\ &= \frac{a+b}{1-aG_X(z)}G_S(z)G_X'(z) \end{aligned}$$

So

$$\left(1-aG_X(z)\right)G_S'(z) = (a+b)G_S(z)G_X'(z)$$

We now feed in the fact that [note: the second sum goes from 1 instead of 0 because $f_0 = 0$]

$$G_S(z) = \sum_{n=0}^{\infty} g_n z^n \qquad G_X(z) = \sum_{n=1}^{\infty} f_k z^k$$

And get

$$\left(1 - a\sum_{\alpha=1}^{\infty} f_\alpha z^\alpha\right)\left(\sum_{\beta=1}^{\infty} \beta g_\beta z^{\beta-1}\right) = (a+b)\left(\sum_{\alpha=0}^{\infty} g_\alpha z^\alpha\right)\left(\sum_{\beta=1}^{\infty} \beta f_\beta z^{\beta-1}\right)$$

Now, equate coefficients of z^{r-1}

$$rg_r - a\sum_{\alpha+\beta=r} \beta f_\alpha g_\beta = (a+b)\sum_{\alpha+\beta=r} \beta g_\alpha f_\beta$$

$$rg_r - a\sum_{\alpha=1}^{r-1}(r-\alpha)f_\alpha g_{r-\alpha} = (a+b)\sum_{\beta=1}^{r} \beta f_\beta g_{r-\beta}$$

And so

$$\begin{aligned} rg_r &= \sum_{\beta=1}^{r}(a\beta + b\beta)f_\beta g_{r-\beta} + \sum_{\alpha=1}^{r-1}(ar - a\alpha)f_\alpha g_{r-\alpha} \\ &= \sum_{\beta=1}^{r-1}(ar + b\beta)f_\beta g_{r-\beta} + (ar+br)f_r g_0 \\ &= \sum_{\beta=1}^{r}(ar + b\beta)f_\beta g_{r-\beta} \end{aligned}$$

Which means that

$$g_r = \sum_{j=1}^{r}\left(a + \frac{bj}{r}\right)f_j g_{r-j}$$

This is our recursion formula for the g, starting from $g_1 = f_1 p_1$.

Let us find a and b when

$$p_n = \frac{e^{-\lambda}\lambda^n}{\left(1-e^{-\lambda}\right)n!}$$

Note that

$$p_{n-1} = \frac{e^{-\lambda}\lambda^{n-1}}{\left(1-e^{-\lambda}\right)(n-1)!}$$

And so

$$\begin{aligned} a + \frac{b}{n} &= \frac{p_n}{p_{n-1}} \\ &= \frac{e^{-\lambda}\lambda^n}{\left(1-e^{-\lambda}\right)n!} \Bigg/ \frac{e^{-\lambda}\lambda^{n-1}}{\left(1-e^{-\lambda}\right)(n-1)!} \\ &= \frac{\lambda}{n} \end{aligned}$$

And so $a = 0$ and $b = \lambda$. Our recursion formula becomes

$$g_r = \sum_{j=1}^{r} \frac{\lambda j}{r} f_j g_{r-j}$$

Starting from $g_1 = f_1 p_1$.

Question 2

2 A portfolio consists of n independent risks. For the i^{th} risk, the number of claims in a year has a Poisson distribution with parameter λ_i and the claims are independent exponentially distributed random variables with mean μ, independent of the number of claims. Let S_i be the total amount claimed in a year for risk i. Find the moment generating function of S_i, and show that $S = S_1 + \ldots + S_n$ has a compound Poisson distribution.

Now suppose that $\lambda_1, \ldots, \lambda_n$ are independent identically distributed random variables with density

$$f(\lambda) = \frac{\alpha^m \lambda^{m-1} e^{-\alpha\lambda}}{(m-1)!}, \quad \lambda > 0$$

for $\alpha > 0$ and $m \in \mathbb{N}$, so that the number of claims for each risk in one year has a mixed Poisson distribution with mixing density $f(\lambda)$. Find the distribution of the total number of claims on the whole portfolio in one year.

Show that the total amount S claimed in one year on the whole portolio has a compound mixed Poisson distribution, and identify the mixing distribution for the Poisson parameter.

The situation is as follows

- Each year has n risks
- Each risk $i \in 1, \cdots, n$ has a number of claims $N_i \sim \text{Po}(\lambda_i)$ per year. The PGF of N_i is

$$G_{N_i}(z) = \mathbb{E}\left(z^{N_i}\right) = e^{\lambda_i(z-1)}$$

- Each claim is of size $X \sim \text{Exp}(\mu)$. The MGF of X is

$$M_X(t) = \mathbb{E}\left(e^{tX}\right) = \frac{\mu}{\mu - t} \qquad t < \mu$$

S_i is the total claim per year for risk i. Clearly, it has a compound distribution. The MGF of S_i is given by

$$\begin{aligned} M_{S_i}(u) = \mathbb{E}\left(e^{uS_i}\right) &= \mathbb{E}\left(\mathbb{E}(e^{uS_i} \mid N_i)\right) \\ &= \mathbb{E}\left(\mathbb{E}(e^{uX_1} e^{uX_2} \cdots \mid N_i)\right) \\ &= \mathbb{E}\left(\mathbb{E}(e^{uX_1} \cdots e^{uX_{N_i}})\right) \\ &= \mathbb{E}\left([e^{uX_1}]^{N_i}\right) && \leftarrow \text{Indep. of } X_i \\ &= \mathbb{E}\left([M_X(u)]^{N_i}\right) \\ &= G_{N_i}\left(M_X(u)\right) \\ &= \exp\left(\lambda_i\left[\frac{\mu}{\mu-u} - 1\right]\right) && \leftarrow \text{Use functions from start of question} \end{aligned}$$

Now, consider $S = S_1 + \cdots + S_n$

$$\begin{aligned} M_S(u) = \mathbb{E}\left(e^{uS}\right) &= \mathbb{E}\left(e^{uS_1} \cdots e^{uS_n}\right) \\ &= \mathbb{E}\left(e^{uS_1}\right) \cdots \mathbb{E}\left(e^{uS_n}\right) && \leftarrow \text{Indep. of } S_i\text{, due to indep. of } X_i \text{ and } \lambda_i \\ &= \exp\left(\lambda_1 \left[\tfrac{\mu}{\mu-u} - 1\right]\right) \cdots \exp\left(\lambda_n \left[\tfrac{\mu}{\mu-u} - 1\right]\right) \\ &= \exp\left(\left[\textstyle\sum_{i=1}^n \lambda_i\right]\left[\tfrac{\mu}{\mu-u} - 1\right]\right) \end{aligned}$$

This is clearly a compound Poisson, with Poisson parameter $\sum_{i=1}^n \lambda_i$.

The total number of claims on the whole portfolio in one year is $N = \sum_{i=1}^n N_i$, and we now have $N_i \sim \mathrm{Po}(\lambda_i)$, where $\lambda \sim f(\lambda_i)$ and $\lambda > 0$.

First, consider each N_i and let x be an integer

$$\begin{aligned} \mathbb{P}\left(N_i = x\right) &= \int_{\lambda=0}^{\infty} \mathbb{P}\left(N_i = x \mid \lambda\right) f(\lambda) \, \mathrm{d}\lambda \\ &= \int_{\lambda=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} \cdot \frac{\alpha^m \lambda^{m-1} e^{-\alpha\lambda}}{(m-1)!} \cdot 1 \, \mathrm{d}\lambda \\ &= \int_{\lambda=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} \cdot \frac{\alpha^m \lambda^{m-1} e^{-\alpha\lambda}}{(m-1)!} \cdot \frac{(x+m-1)!}{(x+m-1)!} \, \mathrm{d}\lambda \\ &= \frac{(x+m-1)!}{(m-1)!} \int_{\lambda=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} \cdot \frac{\alpha^m \lambda^{m-1} e^{-\alpha\lambda}}{(x+m-1)!} \, \mathrm{d}\lambda \\ &= \frac{(x+m-1)!}{(m-1)!} \frac{\alpha^m}{x!} \int_{\lambda=0}^{\infty} \frac{\lambda^{x+m-1} e^{-(\alpha+1)\lambda}}{\Gamma(x+m)} \, \mathrm{d}\lambda \\ &= \frac{(x+m-1)!}{(m-1)!} \frac{\alpha^m}{x!(\alpha+1)^{x+m}} \int_{\lambda=0}^{\infty} \frac{(\alpha+1)^{x+m} \lambda^{x+m-1} e^{-(\alpha+1)\lambda}}{\Gamma(x+m)} \, \mathrm{d}\lambda \end{aligned}$$

At this point, we remember (somehow) that the PDF of $\Gamma(k,\theta)$ is $f(\lambda) = \theta^k \lambda^{k-1} e^{-\theta\lambda} / \Gamma(k)$. The quantity in the integral is in that form, with $k = x + m$ and $\theta = (1+\alpha)^{-1}$. The integral therefore evaluates to 1 and

$$\begin{aligned} \mathbb{P}\left(N_i = x\right) &= \frac{(x+m-1)!}{(m-1)!} \frac{\alpha^m}{x!(\alpha+1)^{x+m}} \\ &= \binom{x+m-1}{x} \alpha^m \left(\alpha+1\right)^{-(x+m)} \\ &= \binom{x+m-1}{x} \left(\tfrac{\alpha}{\alpha+1}\right)^m \left(\tfrac{1}{\alpha+1}\right)^x \end{aligned}$$

This is a negative binomial distribution with parameters

$$p = \tfrac{\alpha}{\alpha+1}$$
$$q = 1 - \tfrac{\alpha}{\alpha+1} = \tfrac{1}{\alpha+1}$$
$$r = m$$

The total number of claims is given by

$$N = N_1 + \cdots + N_n$$

Consider the MGF of N:

$$M_N(t) = \mathbb{E}\left(e^{Nt}\right) = \mathbb{E}\left(e^{N_1 t}\right) \cdots \mathbb{E}\left(e^{N_n t}\right) = M_{N_1}(t) \cdots M_{N_n}(t)$$

Furthermore, since N_i has a negative binomial distribution,

$$M_{N_i}\left(t\right) = \left(\frac{1}{1 - \frac{1}{\alpha+1} e^t}\right)^m$$

And so

$$M_N\left(t\right) = \left(\frac{1}{1 - \frac{1}{\alpha+1} e^t}\right)^{mn}$$

This is also the MGF of a negative binomial, with the same p parameter but with

$$p = \tfrac{\alpha}{\alpha+1}$$
$$r = nm$$

In other words

$$N \sim \text{NegBin}\left(r = nm, p = \tfrac{\alpha}{\alpha+1}\right)$$

This can be viewed as a single Poisson distribution $N \sim \text{Po}(\lambda)$ with its parameter mixed over the following distribution:

$$f(\lambda) = \frac{\alpha^{nm} \lambda^{nm-1} e^{-\alpha\lambda}}{(nm-1)!} \qquad \lambda > 0 \qquad (*)$$

Now, the claims sizes *all* have exponential distribution with parameter μ. And we have just seen that the total number of claims in a year is N, across all categories. Now

$$S = \sum_{k=1}^{n} \sum_{i=1}^{N_k} X_i = \sum_{i=1}^{N} X_i$$

So S does indeed have a compound mixed Poisson distribution, and the mixing distribution is that in equation (*).

Question 3

3 Explain what is meant by a classical risk model with positive premium loading factor.

Assume that the adjustment coefficient is the unique positive solution of

$$M(r) - 1 = (1+\theta)\mu r,$$

where $M(r)$ is the claim size moment generating function, μ is the mean claim size and θ is the premium loading factor. State and prove Lundberg's inequality for the probability of ruin.

Find the adjustment coefficient R when claims are exponentially distributed with mean μ.

Determine whether R is greater or smaller than the adjustment coefficient R_μ for claims that are exactly μ, and comment briefly on the corresponding Lundberg bounds.

In the classical risk model

- The claim sizes $X_1, X_2, \cdots$ are positive random variables.
- The number of claims arriving in $(0, t]$ is $N(t)$, it is independent of the X_i and $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, which means that (a) $N(t) \sim \text{Po}(\lambda t)$ and (b) the times between consecutive arrivals are IID exponential variables with mean $1/\lambda$.
- We assume that premium income is received continuously at a constant rate $c > 0$, and we suppose that at $t = 0$, the company has capital $u \geq 0$.

At time t, the **risk-reserve** is then given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

We note that using the properties of compound Poisson variables

$$\mathbb{E}(U(t)) = u + ct - \lambda\mu t$$

Where $\mu = \mathbb{E}(X_1)$. Thus, the profit the company makes per unit time is given by

$$\frac{\mathbb{E}(U(t)) - u}{t} = c - \lambda\mu$$

If $c > \lambda\mu$, then the expected profit per unit time is positive and we have *positive safety loading*. We write $c = (1+\theta)\lambda\mu$ where $\theta > 0$ is called the premium loading factor.

The probability of ruin is given by

$$\psi(u) = \mathbb{P}\begin{pmatrix} U(t) < 0 \text{ at some time } t \\ \text{given starting capital } u \end{pmatrix}$$

Lundberg's inequality states that

$$\psi(u) \le e^{-ru}$$

Where R, the adjustment coefficient, is given by

$$M_X(r) - 1 = \left(1+\theta\right)\mu r$$

This inequality holds provided that there exists a $z_\infty \in (0,\infty]$ such that $M_X(z) \uparrow \infty$ as $z \to z_\infty$.

We prove this inequality in three steps

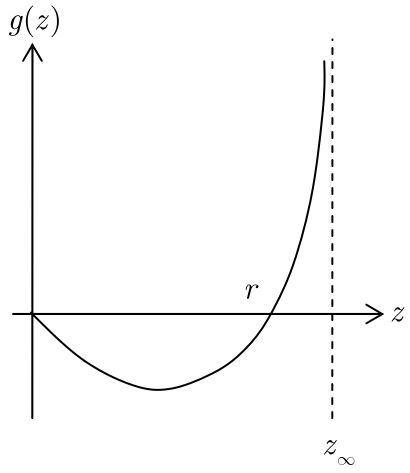
1. **We prove that $M_X(r) - 1 = (1+\theta)\mu r$ has a unique strictly positive solution.**

   Define $g(z) = M_X(z) - 1 - (1+\theta)\mu z$. We would like to show that there is a unique strictly positive solution to $g(z) = 0$.

   First, assume that $z_\infty < \infty$. In that case

   - g is continuous because M is continuous (this is a property of Laplace transforms)
   - $g(0) = M_X(0) - 1 = 1 - 1 = 0$
   - $g'(0) = M_X'(0) - (1+\theta)\mu = -\theta\mu < 0$ because, assuming positive safety loading, $\theta > 0$
   - $g''(0) < 0$ - this is another property of Laplace transforms
   - g tends to ∞ as $z \to z_\infty$

   Together, these imply that g looks like this



Clearly, therefore, there is a single strictly positive solution of $g(z) = 0$.

If $z_\infty = \infty$, we need to make sure that the M term in g grows faster than the r term – otherwise, the function no longer looks as plotted above. To show this, consider that since the variables X are positive,

$$\exists \eta > 0 \text{ s.t. } \mathbb{P}\left(X > \eta\right) = p > 0$$

Now

$$\begin{aligned} M_X(z) &= \mathbb{E}\left(e^{zX}\right) \\ &= p\mathbb{E}\left(e^{zX} \mid X > \eta\right) + (1-p)\mathbb{E}\left(e^{zX} \mid X \leq \eta\right) \\ &\geq p\mathbb{E}\left(e^{zX} \mid X > \eta\right) \\ &\geq pe^{z\eta} \end{aligned}$$

Therefore, M_X is bounded below by an exponential, which clearly grows faster than a simple linear term. Thus

$$g(z) \geq pe^{\eta z} - 1 + (1+\theta)\mu z \to \infty$$

So there is indeed one unique strictly positive solution for r.

2. We define a new quantity, $\psi_n(u)$, such that $\psi_n(u) \leq e^{-Ru}\ \forall n \Rightarrow \psi(u) \leq e^{-Ru}$

The new quantity we define is

$$\psi_n(u) = \mathbb{P}\left(\text{Ruin occurs at or before } n^{\text{th}} \text{ claim}\right)$$

Clearly, $\psi_n(u) \uparrow \psi(u)$ as $n \to \infty$. As such

$$\psi_n(u) \leq e^{-Ru}\ \forall n \Rightarrow \psi(u) \leq e^{-Ru}$$

3. We show that $\psi_n(u) \le e^{-Ru}\ \forall n$

We do this by induction

- $n = 1$ case

  Clearly, ruin can't occur before the first claim. Thus

$$\begin{aligned}\psi_1(u) &= \mathbb{P}\left(\text{Ruin occurs at or before } 1^{\text{st}} \text{ claim}\right)\\ &= \mathbb{P}\left(\text{Ruin occurs at } 1^{\text{st}} \text{ claim}\right)\\ &= \int_0^\infty \mathbb{P}\left(\text{Occurs @ } 1^{\text{st}} \mid 1^{\text{st}} \text{ occurs at } t\right)\lambda e^{-\lambda t}\ \mathrm{d}t\end{aligned}$$

  Consider, however, that at a time t, the risk reserve is $u + ct$. The first claim must exceed this amount for ruin to occur

$$\begin{aligned}\psi_1(u) &= \int_0^\infty \mathbb{P}\left(X_1 > u + ct\right)\lambda e^{-\lambda t}\ \mathrm{d}t\\ &= \int_0^\infty \int_{x=u+ct}^\infty f_X(x)\lambda e^{-\lambda t}\ \mathrm{d}x\,\mathrm{d}t\end{aligned}$$

  Now, note that in the range of the integral, $e^{-r\left(u+ct-x\right)}$ is greater than 1 and so

$$\psi_1(u) \le \int_0^\infty \int_{x=u+ct}^\infty e^{-R(u+ct-x)} f_X(x)\lambda e^{-\lambda t}\ \mathrm{d}x\,\mathrm{d}t$$

  Further note that the integrand is positive, so

$$\begin{aligned}\psi_1(u) &\le \int_0^\infty \int_{x=0}^\infty e^{-r(u+ct-x)} f_X(x)\lambda e^{-\lambda t}\ \mathrm{d}x\,\mathrm{d}t\\ &= \int_0^\infty \lambda e^{-\lambda t} e^{-r(u+ct)} \int_{x=0}^\infty e^{rx} f_X(x)\ \mathrm{d}x\,\mathrm{d}t\\ &= e^{-ru}\int_0^\infty \lambda e^{-(rc+\lambda)t} M_X(r)\ \mathrm{d}t\end{aligned}$$

  r is defined as $M_X(r) = (1+\theta)\mu r + 1$, so

$$\begin{aligned}\psi_1(u) &\le e^{-ru}\int_0^\infty \lambda e^{-(rc+\lambda)t}\left\{(1+\theta)\mu r + 1\right\}\ \mathrm{d}t\\ &= e^{-ru}\int_0^\infty \left\{(1+\theta)\mu\lambda r + \lambda\right\} e^{-(rc+\lambda)t}\ \mathrm{d}t\end{aligned}$$

  Remember that $c = (1+\theta)\mu\lambda$

$$\psi_1(u) \le e^{-ru}\int_0^\infty \left\{rc + \lambda\right\} e^{-(rc+\lambda)t}\ \mathrm{d}t$$

  The integral is simply an exponential density that evaluates to 1, so

$$\psi_1(u) \le e^{-ru}$$

- Inductive step

  Now assume that $\psi_n(u) \le e^{-ru}$, and consider

$$\begin{aligned}\psi_{n+1}(u) &= \mathbb{P}\left(\text{Ruin @ or before } (n+1)^{\text{th}}\right) \\ &= \int_0^\infty \mathbb{P}\begin{pmatrix}\text{Ruin @ or before } (n+1)^{\text{th}} \\ \mid 1^{st} \text{ occurs at } t\end{pmatrix}\lambda e^{-\lambda t}\ \mathrm{d}t\end{aligned}$$

We now split this integral into two options:

- The ruin happening at the first claim (ie: first claim greater than $u + ct$)
- The ruin not happening at the first claim, in which case, after the first claim, we "reset the timer" with capital $u + ct - x_1$

– the ruin happening at the first claim, and the ruin not happening at the first claim:

$$\psi_{n+1}(u) = \int_0^\infty \lambda e^{-\lambda t}\left\{\int_{x_1=u+ct}^\infty f_X(x)\ \mathrm{d}x + \int_{x_1=0}^{u+ct} \psi_n(u+ct-x_1) f_X(x)\ \mathrm{d}x\right\}\ \mathrm{d}t$$

Now:

- In the first situation, $e^{-r(u+ct-x_1)} > 1$
- In the second situation, the inductive hypothesis implies that $\psi_n(u+ct-x_1) \le e^{-r(u+ct-x_1)}$.

As such

$$\begin{aligned}\psi_{n+1}(u) &\le \int_0^\infty \lambda e^{-\lambda t}\left\{\int_{x_1=0}^\infty e^{-r(u+ct-x_1)} f_X(x)\ \mathrm{d}x\right\}\ \mathrm{d}t \\ &\le e^{-ru}\end{aligned}$$

When claims are exponentially distributed,

$$M_X(u) = \left(1 - u\mu\right)^{-1}$$

So

$$\begin{gathered}\frac{1}{1-R\mu} - 1 = (1+\theta)\mu R \\ (1+\theta)(1-R\mu) = 1 \\ 1 - R\mu + \theta - \theta R\mu = 1 \\ \boxed{R = \frac{\theta}{\mu(1+\theta)}}\end{gathered}$$

For claims that are exactly μ, $M_X(u) = \mathbb{E}\left(e^{uX}\right) = e^{u\mu}$, and so

$$e^{R_\mu \mu} - 1 = (1+\theta)\mu R_\mu$$

A Taylor expansion gives

$$R_{\mu}\mu + \frac{1}{2}\left(R_{\mu}\mu\right)^{2} + \cdots = (1+\theta)\mu R_{\mu}$$
$$-\theta\mu R_{\mu} + \frac{1}{2}\mu^{2}R_{\mu}^{2} + \cdots = 0$$

Truncating the Taylor series will result in a value of R_{μ} that is

$$R_{\mu} = \frac{\theta}{\mu\frac{1}{2}}$$

This is clearly larger than R, because $1+\theta > \frac{1}{2}$ this means that the Lundberg bound leads to a generally *lower* probability of ruin in the "fixed claim size" case. This makes sense – the exponential distribution is highly positively skewed, and this implies that it places greater weight on claim sizes *above* the mean than below. Thus, by replacing the exponential distribution with the mean exactly, we are, overall, decreasing claim sizes. The probability of ruin is therefore lower.

[**Note:** I'm not entirely pleased with the argument above, because it's unclear whether truncating the Taylor series over or underestimates R_{μ}, so it seems silly to then use that as a basis for comparison. If anyone can think of a better way, let me know ☺]

Question 4

4 Let Y_i be the number of claims on a group life insurance policy covering m_i lives in year i, $i = 1, \dots, n$. Suppose that

$$\mathbb{P}(Y_i = x) = \binom{m_i}{x} \theta^x (1-\theta)^{m_i - x}, \quad x = 0, \dots, m_i,$$

where $\theta \in (0,1)$ has prior density $f(\theta)$. Let $X_i = Y_i/m_i$ and assume that, given θ, $X_1, \dots, X_n$ are conditionally independent. Suppose θ is estimated by $\hat{\theta} = a_0 + \sum_{i=1}^n a_i X_i$ where $a_0, a_1, \dots, a_n$ are chosen such that $\mathbb{E}_{x,\theta}[(\theta - \hat{\theta})^2]$ is minimised. Show that $\hat{\theta}$ can be written in the form

$$\hat{\theta} = Z \frac{\sum_{i=1}^n m_i X_i}{\sum_{i=1}^n m_i} + (1 - Z)\mathbb{E}[\theta]$$

where you should specify Z.

Now suppose that $f(\theta) = 1$ for $0 < \theta < 1$ and that $n = 2$. Find $\hat{\theta}$, and compare it with the Bayesian estimate of θ with respect to quadratic loss.

We write

$$\hat{\theta} = a_0 + \sum_{i=1}^n a_i X_i$$

We need to choose this estimator to minimise

$$L = \mathbb{E}\left\{\left(\theta - a_0 - \sum_{i=1}^n a_i X_i\right)^2\right\}$$

This implies that

$$\frac{\partial L}{\partial a_0} = \mathbb{E}\left\{\theta - a_0 - \sum_{i=1}^n a_i X_i\right\} = 0 \tag{1}$$

$$\frac{\partial L}{\partial a_r} = \mathbb{E}\left\{X_r\left(\theta - a_0 - \sum_{i=1}^n a_i X_i\right)\right\} = 0 \qquad \forall r \tag{2}$$

First consider $(2) - \mathbb{E}(X_r)(1)$

$$\mathbb{E}\left\{X_r\left(\theta - a_0 - \sum_{i=1}^n a_i X_i\right)\right\} = \mathbb{E}(X_r)\mathbb{E}\left\{\theta - a_0 - \sum_{i=1}^n a_i X_i\right\}$$

$$\mathbb{E}\left(X_r\theta\right) - a_0\mathbb{E}\left(X_r\right) - \sum_{i=1}^n \mathbb{E}\left\{a_i X_r X_i\right\} = \mathbb{E}(X_r)\mathbb{E}(\theta) - a_0\mathbb{E}(X_r) - \mathbb{E}\left(X_r\right)\sum_{i=1}^n \mathbb{E}\left\{a_i X_i\right\}$$

$$\mathbb{E}\left(X_r\theta\right) - \mathbb{E}(X_r)\mathbb{E}(\theta) = \sum_{i=1}^n \mathbb{E}\left\{a_i X_r X_i\right\} - \mathbb{E}\left(X_r\right)\sum_{i=1}^n \mathbb{E}\left\{a_i X_i\right\}$$

$$\mathbb{C}\text{ov}\left(X_r, \theta\right) = \sum_{i=1}^n a_i \mathbb{C}\text{ov}\left(X_r, X_i\right) \tag{3}$$

We now use the conditional variance formula on both sides of (3)

$$\begin{aligned}
\mathbb{C}\text{ov}\left(X_r,\theta\right) &= \mathbb{E}\left[\mathbb{C}\text{ov}\left(X_r,\theta \mid \theta\right)\right] + \mathbb{C}\text{ov}\left[\mathbb{E}\left(X_r \mid \theta\right),\mathbb{E}\left(\theta \mid \theta\right)\right] \\
&= \mathbb{E}\left[\theta\mathbb{C}\text{ov}\left(X_r,1 \mid \theta\right)\right] + \mathbb{C}\text{ov}\left[\theta,\theta\right] \\
&= \mathbb{V}\text{ar}\left(\theta\right) \\
\mathbb{C}\text{ov}\left(X_r,X_i\right) &= \mathbb{E}\left[\mathbb{C}\text{ov}\left(X_r,X_i \mid \theta\right)\right] + \mathbb{C}\text{ov}\left[\mathbb{E}\left(X_r \mid \theta\right),\mathbb{E}\left(X_i \mid \theta\right)\right] \\
&= \mathbb{E}\left[\delta_{ri}\mathbb{V}\text{ar}\left(X_i \mid \theta\right)\right] + \mathbb{C}\text{ov}\left[\theta,\theta\right] \\
&= \mathbb{E}\left[\delta_{ri}\tfrac{1}{m_i^2}\mathbb{V}\text{ar}\left(Y_i \mid \theta\right)\right] + \mathbb{V}\text{ar}\left(\theta\right) \\
&= \mathbb{E}\left[\delta_{ri}\tfrac{1}{m_i^2}m_i\theta(1-\theta)\right] + \mathbb{V}\text{ar}\left(\theta\right) \\
&= \mathbb{E}\left[\delta_{ri}\tfrac{1}{m_i}\theta(1-\theta)\right] + \mathbb{V}\text{ar}\left(\theta\right)
\end{aligned}$$

Feeding this back into (3), we get

$$\begin{aligned}
\mathbb{V}\text{ar}\left(\theta\right) &= \sum_{i=1}^n a_i\left\{\mathbb{E}\left[\delta_{ri}\tfrac{1}{m_i}\theta(1-\theta)\right] + \mathbb{V}\text{ar}\left(\theta\right)\right\} \\
\mathbb{V}\text{ar}\left(\theta\right) &= \frac{a_r}{m_r}\mathbb{E}\left(\theta(1-\theta)\right) + \mathbb{V}\text{ar}(\theta)\sum_{i=1}^n a_i \\
m_r\mathbb{V}\text{ar}\left(\theta\right) &= a_r\mathbb{E}\left(\theta(1-\theta)\right) + m_r\mathbb{V}\text{ar}(\theta)\sum_{i=1}^n a_i \qquad (4)
\end{aligned}$$

Re-arranging (4), we get

$$a_r = \frac{m_r\mathbb{V}\text{ar}\left(\theta\right)}{\mathbb{E}\left(\theta(1-\theta)\right)}\left\{1-\sum_{i=1}^n a_i\right\} \qquad (5)$$

Summing (4) from 1 to n, we get

$$\sum_{i=1}^n a_i = \frac{m_+}{m_+ + \frac{\mathbb{E}\left(\theta(1-\theta)\right)}{\mathbb{V}\text{ar}(\theta)}} \qquad (6)$$

Where $m_+ = \sum_{i=1}^n m_i$. Feeding (6) into (5), we get

$$a_r = \frac{m_r\mathbb{V}\text{ar}\left(\theta\right)}{\mathbb{E}\left(\theta(1-\theta)\right)}\left\{1-\frac{m_+}{m_+ + \frac{\mathbb{E}\left(\theta(1-\theta)\right)}{\mathbb{V}\text{ar}(\theta)}}\right\}$$

$$\boxed{a_r = \frac{m_r}{m_+}\left\{\frac{m_+\mathbb{V}\text{ar}(\theta)}{m_+\mathbb{V}\text{ar}(\theta)+\mathbb{E}\left(\theta(1-\theta)\right)}\right\}}$$

Going all the way back to (1), we get

$$\begin{aligned}
\mathbb{E}(\theta) - a_0 - \sum_{i=1}^n a_i\mathbb{E}(X_i) &= 0 \\
a_0 &= \mathbb{E}(\theta) - \sum_{i=1}^n a_i\tfrac{1}{m_i}\mathbb{E}(Y_i) \\
a_0 &= \mathbb{E}(\theta)\left\{1-\sum_{i=1}^n a_i\right\}
\end{aligned}$$

Feeding (6) into this, we get

$$\mathbb{E}(\theta) - a_0 - \sum_{i=1}^{n} a_i \mathbb{E}(X_i) = 0$$

$$a_0 = \mathbb{E}(\theta) - \sum_{i=1}^{n} a_i \tfrac{1}{m_i} \mathbb{E}(Y_i)$$

$$\boxed{a_0 = \mathbb{E}(\theta)\left\{1 - \frac{m_+ \mathbb{V}\text{ar}(\theta)}{m_+ \mathbb{V}\text{ar}(\theta) + \mathbb{E}\Big(\theta(1-\theta)\Big)}\right\}}$$

Feeding this all into $\hat{\theta} = a_0 + \sum_{i=1}^{n} a_i X_i$, we obtain

$$\hat{\theta} = \mathbb{E}(\theta)\left\{1 - \tfrac{m_+ \mathbb{V}\text{ar}(\theta)}{m_+ \mathbb{V}\text{ar}(\theta) + \mathbb{E}\left(\theta(1-\theta)\right)}\right\} + \sum_{i=1}^{n}\left\{\tfrac{m_+ \mathbb{V}\text{ar}(\theta)}{m_+ \mathbb{V}\text{ar}(\theta) + \mathbb{E}\left(\theta(1-\theta)\right)}\right\}\frac{m_i}{m_+} X_i$$

$$\hat{\theta} = \left\{\tfrac{m_+ \mathbb{V}\text{ar}(\theta)}{m_+ \mathbb{V}\text{ar}(\theta) + \mathbb{E}\left(\theta(1-\theta)\right)}\right\}\frac{\sum_{i=1}^{n} m_i X_i}{m_+} + \left\{1 - \tfrac{m_+ \mathbb{V}\text{ar}(\theta)}{m_+ \mathbb{V}\text{ar}(\theta) + \mathbb{E}\left(\theta(1-\theta)\right)}\right\}\mathbb{E}(\theta)$$

Precisely as required, with

$$Z = \frac{m_+ \mathbb{V}\text{ar}(\theta)}{m_+ \mathbb{V}\text{ar}(\theta) + \mathbb{E}\Big(\theta(1-\theta)\Big)}$$

Now, if $f(\theta) = \mathbb{I}_{\theta \in (0,1)}$, then

- $\mathbb{E}(\theta) = \frac{1}{2}$
- $\mathbb{V}\text{ar}(\theta) = \frac{1}{12}$
- $\mathbb{E}\Big(\theta(1-\theta)\Big) = \mathbb{E}\Big(\theta - \theta^2\Big) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$
- $Z = \dfrac{m_+ \frac{1}{12}}{m_+ \frac{1}{12} + \frac{1}{6}} = \dfrac{m_+}{m_+ + 2}$

And so

$$\hat{\theta} = \frac{m_+}{m_+ + 2}\frac{\sum_{i=1}^{n} m_i X_i}{m_+} + \left\{1 - \frac{m_+}{m_+ + 2}\right\}\frac{1}{2}$$

$$\hat{\theta} = \frac{1 + \sum_{i=1}^{n} m_i X_i}{m_+ + 2}$$

$$\boxed{\hat{\theta} = \frac{1 + \sum_{i=1}^{n} Y_i}{2 + \sum_{i=1}^{n} m_i}}$$

In terms of exact credibility; the posterior is given by (once again, we write $y_+ = \sum_{i=1}^n y_i$, and similarly for other quantities):

$$\begin{aligned}\pi(\theta \mid \boldsymbol{y}) &\propto f(\theta)\mathbb{P}\left(\boldsymbol{Y} = \boldsymbol{y} \mid \theta\right) \\ &= \mathbb{P}\left(\boldsymbol{Y} = \boldsymbol{y} \mid \theta\right)\mathbb{I}_{\theta\in(0,1)} \\ &\propto \prod_{i=1}^n \theta^{y_i}(1-\theta)^{m_i - y_i}\mathbb{I}_{\theta\in(0,1)} \\ &= \theta^{y_+}(1-\theta)^{m_+ - y_+}\mathbb{I}_{\theta\in(0,1)} \\ &\sim \text{Beta}\left(y_+ + 1, m_+ - y_+ + 1\right)\end{aligned}$$

And the Bayesian estimate, with respect to quadratic loss, is therefore simply the mean of the beta distribution, given by

$$\mathbb{E}\left(\theta \mid \boldsymbol{y}\right) = \frac{y_+ + 1}{2 + m_+}$$

$$\boxed{\mathbb{E}\left(\theta \mid \boldsymbol{y}\right) = \frac{1 + \sum_{i=1}^n Y_i}{2 + \sum_{i=1}^n Y_i}}$$

In this case, it's exactly the same. Thus, in this particular case, exact Bayes' credibility is possible, and the resulting estimate is identical to the Buhlman credibility estimate.