Poisson's Equation

Physical Origins

- **Poisson's Equation** is

$$\nabla^2 \Phi = \sigma(\boldsymbol{x})$$

Sometimes, $\sigma(\boldsymbol{x}) = 0$, in which case we have **Laplace's Equation**.

- ***Diffusion***
  - Consider some quantity Φ which **diffuses** (eg: *heat*, *concentration of a dilute chemical*, etc…)
  - There is a corresponding **flux**, **F** – the **amount crossing unit area per unit time**. ***Experimentally***, this is given by

$$\boldsymbol{F} = -k\nabla\Phi$$

    Notes:
    - Note the **minus sign** – the flux is **directed** towards the area of **lower concentration**.
    - k is called the **diffusivity** in the case of a chemical, and the **coefficient of heat conductivity** in the case of temperature.
  - We then note that if V is a volume bounded by a surface S, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\iiint_V \Phi \,\mathrm{d}V\right] = -\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S = \iint_S k\nabla\Phi\cdot\boldsymbol{n}\,\mathrm{d}S$$

    And so by the Divergence Theorem

$$\iiint_V \frac{\mathrm{d}\Phi}{\mathrm{d}t}\,\mathrm{d}V = \iiint_V \nabla\cdot(k\nabla\Phi)\,\mathrm{d}V$$

    But since this must be true for **all** volumes:

$$\boxed{\frac{\mathrm{d}\Phi}{\mathrm{d}t} = k\nabla^2\Phi + S(\boldsymbol{x})}$$

    Where S corresponds to **sources** or **sinks**.
  - In the case of temperature, things are more complicated, because $-k\nabla^2 T$ gives the rate of

change of **heat**, which needs to be related to T using $H = \rho c T$.

- In the **steady state**, $\mathrm{d}\Phi/\mathrm{d}t = 0$, and we therefore recover **Laplace's Equation**.

- ***Electrostatics***
  - One of **Maxwell's Equations** gives

$$\nabla^2\Phi = -\rho/\varepsilon_0$$

  - Another is

$$\nabla \times \boldsymbol{B} = 0$$

So there exists a **magnetostatic potential** ψ such that $\boldsymbol{B} = -\mu_0\nabla\psi$ and $\nabla^2\psi = 0$.

- ***Gravitation***
  - Consider a **mass distribution** $\rho(\boldsymbol{x})$ – there is a **corresponding gravitational field** $\mathbf{F}(\boldsymbol{x})$ that can be expressed in terms of a **potential** $\Phi(\boldsymbol{x})$.
  - If an **arbitrary volume** $\boldsymbol{V}$ is bounded by a **surface** $\boldsymbol{S}$ containing a total mass $M_V = \iiint_V \rho(\boldsymbol{x})\,\mathrm{d}V$, the **flux** of the **field** through S is $-4\pi G M_V$.
  - Therefore

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S = -4\pi G M_V$$

$$-\iint_S \nabla\Phi\cdot\boldsymbol{n}\,\mathrm{d}S = -4\pi G\iiint_V \rho(\boldsymbol{x})\,\mathrm{d}V$$

$$\iiint_V \nabla\cdot(\nabla\Phi)\,\mathrm{d}V = 4\pi G\iiint_V \rho(\boldsymbol{x})\,\mathrm{d}V$$

$$\boxed{\nabla^2\Phi = 4\pi G\rho}$$

Separation of Variables – Laplace's Equation

- ***Plane polar coordinates***
  - In **plane polars**, if we know that the solution is **axisymmetric** (ie: Φ does **not** depend on ϕ) Laplace's Equation becomes:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial \theta^2}=0$$
  - Consider a solution of the form
$$\Phi(r,\theta)=R(r)\Theta(\theta)$$
  - This becomes:
$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{\Theta''}{\Theta}$$
  - Each side must now be equal to a constant…
  - **Angular part**
$$\Theta''=-\lambda\Theta$$
$$\Theta=\begin{cases}A+B\theta & \lambda=0\\ A\cos\left(\theta\sqrt{\lambda}\right)+B\sin\left(\theta\sqrt{\lambda}\right) & \lambda\neq 0\end{cases}$$
    To obtain a sensible solution, replacing $\theta\to\theta+2\pi$ should make no difference to $\nabla\Phi$. This can only happen if $\Theta'(\theta+2\pi)=\Theta'(\theta)$. Therefore
    - ***Either*** $\lambda=0$
    - ***Or***
$$\left.\begin{aligned}\cos 2\pi\sqrt{\lambda}=1\\ \sin 2\pi\sqrt{\lambda}=0\end{aligned}\right\}\Rightarrow\sqrt{\lambda}=n\quad \lambda>0$$
    Therefore:
$$\Theta=\begin{cases}A+B\theta & n=0\\ A\cos(n\theta)+B\sin(n\theta) & n\neq 0\end{cases}$$
  - **Radial part**
$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\lambda=n^2$$
$$r^2R''+rR'-n^2R=0$$

Substitute $r = e^u$ to get

$$R = \begin{cases} C + D\ln r & n = 0 \\ Cr^n + Dr^{-n} & n \neq 0 \end{cases}$$

- **Solution**

  Therefore, the solution is

  $$\Phi = R\Theta = \begin{cases} (C + D\ln r)(A + B\theta) & n = 0 \\ (Cr^n + Dr^{-n})(A\cos n\theta + B\sin n\theta) & n \neq 0 \end{cases}$$

  The combination $\theta \ln r$ doesn't satisfy the periodicity required, so we exclude it.

  The general solution, therefore, including a superposition of solutions, is:

$$\boxed{\Phi = A_0 + B_0\theta + C_0 \ln r + \sum_{n=1}^{\infty}(A_n r^n + C_n r^{-n})\cos n\theta + \sum_{n=1}^{\infty}(B_n r^n + D_n r^{-n})\sin n\theta}$$

- ***Note***: we have required $\nabla\Phi$ to be **periodic**, because it is always a **physical quantity**. However, sometimes, Φ itself is also a **physical quantity**, and also needs to be **periodic**. In such a case, $\boldsymbol{B_0 = 0}$.

- ***Spherical polar coordinates***
  - In **plane polars**, Laplace's Equation becomes:

    $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0$$

  - Consider a solution of the form

    $$\Phi(r,\theta) = R(r)\Theta(\theta)$$

  - This becomes [***note***: leave as it is!]

    $$\frac{1}{R}\left(r^2 R'\right)' = -\frac{1}{\Theta\sin\theta}\left(\Theta'\sin\theta\right)'$$

  - Each side must now be equal to a constant…
  - **Angular part**

    $$\left(\Theta'\sin\theta\right)' = -\lambda\Theta\sin\theta$$

    Let $\zeta = \cos\theta$, and use

$$\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = -\sin\theta\frac{\mathrm{d}}{\mathrm{d}\zeta}$$
$$\sin\theta = \sqrt{1-\zeta^2}$$

The equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left((1-\zeta^2)\frac{\mathrm{d}\Theta}{\mathrm{d}\zeta}\right) + \lambda\Theta = 0$$

This is **Legendre's Equation**. For **well behaved solutions** at $\zeta = \pm 1$, we need

$$\lambda = n(n+1) \qquad n \geq 0$$

And the solution becomes:

$$\Theta = CP_n(\zeta) = CP_n(\cos\theta)$$

- **Radial part**

$$\left(r^2 R'\right)' = \lambda R$$
$$r^2 R'' + 2rR' - n(n+1)R = 0$$

Using similar methods as above, we get

$$R = Ar^n + Br^{-n-1}$$

- **Solution**

The **general solution** is therefore

$$\boxed{\Phi(r,\theta) = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta)}$$

Uniqueness Theorems

- Consider the following Poisson Equation

$$\nabla^2\Phi = \sigma(\boldsymbol{x})$$

We can show that **any** solution Φ we find to this problem **in** $\boldsymbol{V}$ subject to **either Neumann *or* Dirichlet** boundary conditions **on** $\boldsymbol{S}$ is **unique**.

- Let's assume that there *are* two different solutions, Φ_1 and Φ_2, and let

$$\Psi = \Phi_1 - \Phi_2$$
$$\nabla^2\Psi = \nabla^2\Phi_1 - \nabla^2\Phi_2 = 0$$

And, depending on what conditions we have, **either**

$$\Psi_{\text{on } S} = 0$$
$$\mathrm{d}\Psi/\mathrm{d}n\big|_{\text{on } S} = 0$$

- Now, consider

$$\nabla \cdot (\Psi \nabla \Psi) = |\nabla \Psi|^2 + \Psi \nabla^2 \Psi$$

$$\iiint_V |\nabla \Psi|^2 + \overbrace{\Psi \nabla^2 \Psi}^{\substack{=0,\text{ because} \\ \nabla^2\Psi=0}} \, \mathrm{d}V = \overbrace{\iiint_V \nabla \cdot (\Psi \nabla \Psi) \, \mathrm{d}V}^{\text{Use the divergence theorem}}$$

$$\iiint_V |\nabla \Psi|^2 \, \mathrm{d}V = \iint_S \Psi \nabla \Psi \cdot \boldsymbol{n} \, \mathrm{d}S$$

$$\iiint_V |\nabla \Psi|^2 \, \mathrm{d}V = \overbrace{\iint_S \Psi \frac{\partial \Psi}{\partial n} \mathrm{d}S}^{\substack{\text{One of those two must be 0} \\ \text{(depending on N. or D. BCs)}}}$$

$$\iiint_V |\nabla \Psi|^2 \, \mathrm{d}V = 0$$

- Finally, since $|\nabla \Psi|^2$ can *never* be negative, we must have $\nabla \Psi = 0$. In other words $\Phi_1 - \Phi_2$ is **constant** in V.
  - If **Dirichlet BCs** are given, then $\Phi_1 = \Phi_2$ **somewhere** on S, and therefore $\Phi_1 = \Phi_2$ **everywhere** on S.
  - If **Neumann BCs** are given, the solutions can differ by a constant.

Minimum & Maximum Properties of Laplace's Equation

- Consider Φ that satisfies

$$\nabla^2 \Phi = 0$$

in a **volume *V*** with a **surface *S***.

- Let ***m*** be the **minimum value of Φ on *S***, and let ***M*** bet the **maximum value**.
- Then
  - ***Either m = M***, and Φ is constant **everywhere**.
  - ***Or*** $m < \Phi < M$ in $V - S$
- For a **partial proof**, imagine a **maximum** somewhere **within *V***. The point must be **stationary**, so

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z}$$

However, for it to be a **maximum**, we need

$$\frac{\partial^2 \Phi}{\partial x^2} < 0 \qquad \frac{\partial^2 \Phi}{\partial y^2} < 0 \qquad \frac{\partial^2 \Phi}{\partial z^2} < 0$$

Which is **impossible** since $\nabla^2 \Phi = 0$. This is only a **partial** proof, because it is possible to have a **maximum** with

$$\frac{\partial^2 \Phi}{\partial x^2} = 0 \qquad \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad \frac{\partial^2 \Phi}{\partial z^2} = 0$$