Ordinary Differential Equations

Introduction

- The **general linear first-order ODE**
$$y'(x) + p(x)y(x) = f(x)$$
can be solved using an integrating factor $g = e^{\int p\,\mathrm{d}x}$, to obtain the general solution:
$$y = \frac{1}{g}\int gf\,\mathrm{d}x$$
- A **general linear second order ODE** takes the form
$$a(x)y'' + b(x)y' + c(x)y = f(x)$$
- We look at how to solve general **homogenous** equations of this form, with $f(x) = 0$. **Inhomogeneous** forms can be solved using Green's Functions.

The Wronskian

- If we divide through by the coefficient of y'', we get the equation in standard form
$$y'' + p(x)y' + q(x)y = 0$$
- If we suppose that y_1 and y_2 are solutions, then they are linearly independent if
$$Ay_1(x) + By_2(x) = 0 \Leftrightarrow A = B = 0$$
- If they are linearly independent, then the **general solution** of the ODE is
$$y(x) = Ay_1(x) + By_2(x)$$
- The **Wronskian** $\boldsymbol{W}$ of two solutions y_1 and y_2 of a second-order equation is
$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

- Now, let's imagine that $Ay_1(x) + By_2(x) = 0$. Differentiating $Ay_1'(x) + By_2'(x) = 0$. Therefore, in matrix form, the condition for linear independence is
$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
As such, if the solutions are **linearly dependent**, and the above is true even though A and B aren't 0, then we must have
$$W = 0$$
- Therefore, the solutions y_1 and y_2 are only linearly independent if $W[y_1, y_2] \neq 0$.
- To calculate W, consider
$$\begin{aligned} W' &= y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1' \\ &= y_1 y_2'' - y_2 y_1'' \\ &\overset{\substack{\text{Using the} \\ \text{differential eq}}}{=} y_1\left(-p y_2' - q y_2\right) - y_2\left(-p y_1' - q y_1\right) \\ &= -pW \end{aligned}$$
So solve the first-order ODE
$$W = e^{-\int p \, \mathrm{d}x}$$
Notes:
    - The indefinite integral involves an arbitrary addition constant, so W involves an arbitrary multiplicative constant.
    - If p is **integrable** and $W \neq 0$ for **one value** of x, then $W \neq 0$ for **all** values of x. So linear independence only needs be checked for one value.

Finding a Second Solution

- Suppose that one solution y_1 is known, we can find a second solution y_2 using the original definition of W

$$y_1 y_2' - y_2 y_1' = W$$

$$\frac{y_2'}{y_1} - \frac{y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2}{y_1}\right) = \frac{W}{y_1^2}$$

$$y_2 = y_1 \int \frac{W}{y_1^2}\,\mathrm{d}x$$

  Notes:

  - The indefinite integral involves an arbitrary additive constant, because any amount of y_1 can beaded to y_2.
  - This expression provides the **general** solution of the ODE.

Series Solutions – Introduction

- If we consider a homogeneous linear second-order ODE in standard form

  $$y'' + p(x)y' + q(x)y = 0$$

  A point $x = x_0$ is an **ordinary point** of the ODE if

  $p(x)$ and $q(x)$ are both **analytic** at $x = x_0$

  Otherwise, it is a **singular point**.
- A **singular point** at $x = x_0$ is **regular** if

  $(x - x_0)p(x)$ *and* $(x - x_0)^2 q(x)$ **both** analytic at $x = x_0$
- To find the behaviour as $x \to \infty$, simply replace x by $1/x$, and find the behaviour as $x \to 0$.

Series Solutions about an ordinary point

- If $x = x_0$ is an ordinary point, the ODE has two independent solutions that are also analytic of the form

  $$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

  To find the coefficients, a_n, we simply need to:
  - Substitute this series into the differential equation.
  - Re-label constants (and therefore change the limits of the series) so that only terms in x^n remain.
  - Equate coefficients of x^n to obtain a recurrence relation for a_n.
- Notes
  - An even solution is obtained by choosing $a_0 = 1$ and $a_1 = 0$, and vice-versa for an odd solution. These are clearly linearly independent.
  - The radius of convergence of these solutions are the distance to the singular points of the function.

Series Solutions about a regular singular point

- If $x = x_0$ is a regular singular point, *Fuch's Theorem* guarantees that there is at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma} \qquad a_0 \neq 0$$

  This **Frobenius Series** is a Taylor series (and therefore analytic) if and only if σ is a non-negative integer. The condition is required to define σ uniquely.
- To solve a differential equation at a regular singular point, simply assume a solution of this from and substitute into the differential equation. After re-labelling indices, compare the coefficients of $x^{n+\sigma}$ (possibly obtaining different equations for different values of n).
  - One of the equations the provides the **indicial equation** (given that $a_0 \neq 0$), which can be used to find σ.
  - Another equation can be used to find a_1.
  - A last one can be used to find a recurrence relation.
- Another way, in general, of finding the **indicial equation** (especially if the singularity is not at $z = 0$) is to write the differential equation as

$$y'' + \frac{s(z)}{(z - z_0)} y' + \frac{t(z)}{(z - z_0)^2} y = 0 \text{y}$$

  and then to feed the Frobenius Series into the equation. We can then divide by $z^{\sigma - 2}$ and let $z = z_0$ to obtain the indicial equation.
- In many cases, two solutions will be obtained, because of two roots of σ. However, in some cases, the recurrence relations will fail, or there'll only be one root of σ. In general

- If the roots of the indicial equation are equal, there's only one solution of the form above.
- If they differ by an integer, the recurrence relation will usually fail for the smaller value of $\mathrm{Re}(\sigma)$. [This is to do with the fact that since the difference is an integer, y_1/y_2 is constant and therefore the two solutions are linearly dependent].
- Otherwise, there are two solutions of this form.

* If the roots are equal or differ by an integer, the second solution is of the form

$$y = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\sigma_2} + c y_1 \ln(x - x_0)$$

The coefficients can be determined by substituting into the other ODE and comparing coefficients of $(x - x_0)^n$ and $(x - x_0)^n \ln(x - x_0)$. In exceptional cases, c may vanish.
* Alternatively, y_2 can be found using the Wronskian method.
* Again, the radius of convergence of the series is the distance from the point of expansion to the nearest singular point.

