Dynamics – Rigid Body Dynamics

Introduction

- A rigid body is a many-particle system in which the distance between particles is fixed. The location of all particles is described by 6 coordinates – 3 spatial and 3 angular.
- The velocity is determined by \( v \), the velocity of the CoM and \( \omega \), the angular velocity.
- The basic two equations of angular motion are

\[
M \ddot{R} = F_0
\]

The centre of mass moves as if it were a single particle under the action of a force \( F_0 \).

\[
\dot{J} = G_0
\]

The rate of change of angular momentum is equal to the total applied couple.

- Other basic equations:
  - The velocity \( v \) of a particle at a distance \( r \) from an axis around which a rotation at speed \( \omega \) is happening is
    \[
    v = \omega \times r
    \]
  - For similar reasons:
    \[
    \frac{dJ}{dt} = \omega \times J
    \]
  - Angular speeds are additive. To if frame 1 is rotating with \( \omega_{1\text{ wrt } 2} \) with respect to frame 2, which is rotating with \( \omega_{2\text{ wrt } 3} \) with respect o frame 3, then
    \[
    \omega_{1\text{ wrt } 3} = \omega_{1\text{ wrt } 2} + \omega_{2\text{ wrt } 3}
    \]

Relating \( J \) and \( \omega \)

- If the body is rotating at \( \omega \), the total angular momentum is given by


\[ J = \sum r \times p \]
= \[ \sum r \times m(\omega \times r) \]
= \[ \sum m[r^2 \omega - (\omega \cdot r)r] \]
= \[ \sum m[r^2 \omega - (\omega_x x + \omega_y y + \omega_z z)r] \]

In detail

\[
J = \begin{bmatrix}
\sum m(y^2 + z^2) & -\sum mxz & -\sum mxy \\
-\sum mxy & \sum m(x^2 + z^2) & -\sum myz \\
-\sum mxz & -\sum myz & \sum m(x^2 + y^2)
\end{bmatrix}
\omega
\]

\[ J = I\omega \]

[The non-diagonal elements are fairly easy to derive. The diagonal ones should actually have \(x^2 + y^2 + z^2\), because one of the terms is always knocked out by the second term in the sum]. In other words, \( J \) is proportional to \( \omega \), but not necessarily parallel to it.

- The off-axes elements are rather hard to understand – they correspond to the fact that looking at a particle at a given instant, it’s impossible to tell exactly around which axis it’s moving.
- Also, we can find the kinetic energy

\[
T = \sum \frac{1}{2} m[(\omega \times r) \cdot (\omega \times r)]
= \sum \frac{1}{2} m[\omega \cdot r \times (\omega \times r)]
\]

\[ T = \frac{1}{2} \omega \cdot J \]

- The couple is then given by

\[ G = \dot{J} = \omega \times J \]

- Note that \( I \) must be specified with its origin and with its set of axes.

Properties of \( I \)

- \( I \) is a symmetric tensor. It therefore has three real eigenvalues and three perpendicular eigenvectors.
- With respect to the eigenvector basis:
The eigenvector axes are called the principal axes, and the Is are called the principal moments of inertia.

An alternative way to think of this is that the principal axes are ones around which objects are “happy” to rotate without any torque being applied.

In $\omega$-space, surfaces of constant $T$ form an ellipsoid, with axes of length $\propto I^{-1/2}_a$. Also, in $\omega$-space:

$$\nabla T = I_\alpha \omega_\alpha = J$$

So $J$ is perpendicular to surfaces of constant $T$ at $\omega$.

We can classify the principal axes as follows:

- Spherical tops – all the $I$s are equal, and $J = I \omega$, with $I$ scalar. The body is isotropic with the same $I$ about any axis (eg: sphere, cube).

- Symmetrical tops – $I_1 = I_2 \neq I_3$. $e_3$ axis is unique, but $e_1$ and $e_2$ are any two mutually perpendicular vectors perpendicular to $e_3$ (eg: lens, cigar).

- Asymmetrical tops – all $I$s different, and axes are unique.

Consider any two $I$s:

$$I_1 + I_2 = \sum m(y^2 + z^2 + x^2 + z^2) = I_3 + 2 \sum mz^2 \geq I_3$$

So no $I$ can be larger than the sum of the other two. Furthermore, if $z = 0$ for every particle (ie: if we have a lamina), then

$$I_3 = I_1 + I_2$$

Consider an axis at a distance $a$ away from a principal axis and parallel to it, and let $r$ be the distance of each particle from the principal axis.

Then:

$$I = \sum m(r + a) \cdot (r + a) = I_0 + Ma^2 + 2 \left( \sum mr \right) \cdot a = I_0 + Ma^2$$

=0 when $r$ measured relative to C of M
This is the **Parallel Axis Theorem**, where each vector is considered to be a **projection** in a plane **perpendicular** to the **axes**.

## Two Basic Problems

- **You whack it** – what happens? Steps for solution:
  - Define principal axes with a sensible origin.
  - Calculate an expression for \( J \) in terms of the impulse:
    \[
    J = \int \tau \, dt = \int r_B \times F \, dt = r_B \times \int F \, dt = r_B \times P
    \]
    Where \( B \) is the point at which the whack occurred, and \( r_B \) can be taken out of the integral because the whack is assumed to be instantaneous.
  - Work out an expression for \( J \) in terms of \( \omega \), using the moments of inertia.
  - Equate the two expressions for \( J \).
  - Work out the motion of the CM using standard linear mechanics.
  - **NOTE**: The obvious origin to use is the CM, but other origins can be used subject to the provisos above for using \( \tau = J \). So a pivot, for example, is fine to use.

- **You apply a torque** – what’s the **frequency of rotation**?
  - Define principal axes with a sensible origin (eg: the CM – see above).
  - Find an expression for \( \omega \) in these axes (with unknown magnitude), and find a corresponding expression for \( J \), using the principal moments of inertia.
  - Find \( \frac{dL}{dt} = \omega \times L \).
  - Calculate the **torque** (\( = r \times F \)) and equate it with \( \frac{dL}{dt} \).

## Free Motion – Euler’s Equation

- **Free precession** is a situation in which \( F = 0 \) and \( G = 0 \). In such a case, \( J \) is constant. \( \omega \) is constant if \( J \) is along one of the **principal axes**, but otherwise, it will change **direction**, and perhaps even **magnitude**.

- We use the **Euler Equations** to analyse this problem.
• The rate of change of angular momentum vector in the principal-axes frame (which is rotating around with the body) and the lab frame are related by
\[
\frac{dJ}{dt}_{\text{lab}} \neq \frac{dJ}{dt}_{\text{PA}} + \omega \times J
\]

• Now, let’s assume that a couple \( \mathbf{G} \) is being applied in the lab frame. We know that
\[
\mathbf{G} = \frac{dJ}{dt}_{\text{lab}}
\]
Therefore, using the equations above:
\[
\frac{dJ}{dt}_{\text{PA}} = \frac{dJ}{dt}_{\text{lab}} + \omega \times J
\]

• Finally, we note that in the principal axes frame, \( \mathbf{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) \). Therefore, casting both sides of this equation into the principal axes frame only
\[
\tau_i = I_i\omega_i + (I_3 - I_1)\omega_1\omega_2
\]
And similarly with any cyclic permutation of indices.

• A few notes:
  o All the quantities in this equation are measured with respect to the body frame (which is moving). This is the advantage of these equations – all we have to consider is the forces that the body “feels”.
  o The two terms of the RHS refer to two types of ways \( \mathbf{J} \) can change – because it can change in the body frame and also because the body frame is itself rotating.

Free Motion – Examples

• **Free Symmetric Top**
  o For a symmetrical top \( I_1 = I_2 = I \) which is free in space (ie: no torque) the Euler Equations become
\[
I\dot{\omega}_1 + (I_3 - I)\omega_2\omega_3 = 0
\]
\[
I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3 = 0
\]
\[
I_3\dot{\omega}_3 = 0
\]
  o The last equation implies \( \omega_3 \) is constant. Let’s define
\[ \Omega = \frac{I_3 - I}{I} \omega_3 \]

Then the general solution of the first two equations becomes:
\[
\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \cos[\Omega t + \phi] \\ \cos[\Omega t + \phi] \end{pmatrix}
\]

- **Interpretation from the body frame**
  - In the body frame, \( \omega_1 \) and \( \omega_2 \) seem to form a **circle** in the \( x-y \) plane, with **frequency** \( \Omega \). How **high** that circle is depends on \( \omega_3 \).
  - \( L \) could be **above** \( \omega \) (if \( I_3 > I \) – an **oblate top**) or **below** \( \omega \) (if \( I_3 < I \) – a **prolate top**).

- **Interpretation from the fixed lab frame**
  - In that case, the Euler Equations are useless, because they deal with the body frame, so we express things from scratch, but in **terms of the body frames**:
    \[
    \omega = (\omega_1 \hat{x}_1 + \omega_2 \hat{x}_2) + \omega_3 \hat{x}_3 \\
    L = I(\omega_1 \hat{x}_1 + \omega_2 \hat{x}_2) + I_3 \omega_3 \hat{x}_3 \\
    \downarrow
    \omega = \frac{L}{I} - \Omega \hat{x}_3
    \]
    With \( \Omega \) defined as above.
  - This **linear relationship** between \( \omega \), \( L \) and \( \hat{x}_3 \) implies that they are in the **same plane**.
  - Furthermore, the **rate of change of** \( \hat{x}_3 \) is \( \omega \times \hat{x}_3 \), because it **only** changes as a result of the rotation. So
    \[
    \frac{d\hat{x}_3}{dt} = \left( \frac{L}{I} - \Omega \hat{x}_3 \right) \times \hat{x}_3 = \left( \frac{L}{I} \right) \times \hat{x}_3
    \]
    This is equivalent to \( \hat{x}_3 \) rotating at a frequency \( L/I \).
  - It turns out that we can interpret \( \omega \) as follows
    \[
    \omega = -\frac{L}{I} - \Omega \hat{x}_3
    \]
    - **Heavy Symmetric Top**
      - Here, we must define the **Euler angles** as follows
The total angular velocity is then given by

\[
\mathbf{\omega} = \dot{\psi}\hat{x}_3 + \dot{\theta}\hat{x}_1 + \phi\mathbf{z}
\]

Which can be expressed in terms of the body-frames only:

\[
\mathbf{\omega} = \dot{\psi}\hat{x}_3 + \dot{\theta}\hat{x}_1 + \phi(\hat{x}_3 \cos \theta + \hat{x}_2 \sin \theta)
\]

\[
\mathbf{\omega} = (\dot{\psi} + \phi \cos \theta)\hat{x}_3 + (\dot{\phi} \sin \theta)\hat{x}_2 + \dot{\theta}\hat{x}_1
\]
Dynamics – Normal Modes

Introduction

- A normal mode of a system is an oscillation that has a single frequency.
- All the more general oscillations of the system can be expressed as superpositions of these normal modes.

General approach

- Consider a system defined by generalised coordinates $q_i$ and acted on by forces $F_i$, moving in a potential well $U(x)$, and moving elastically.
- The kinetic energy, $T$, is then given by

$$T = \frac{1}{2} \sum \sum m_i \left| \dot{\xi}_j(q_i) \right|^2$$

Where $\sum \xi_j(q_i)$ is the Cartesian coordinate of the $j^{th}$ part of the system, taken about an equilibrium, where all the $\xi_j$ are 0. Expanding about that equilibrium:

$$\sum_i \xi_j(q_i) = \sum_i \xi_j(q_i,eq) + \frac{\partial \xi_j}{\partial q_i,eq} q_i + \cdots$$

$$\sum_i \dot{\xi}_j(q_i) \approx \sum_i \left( \frac{\partial \xi_j}{\partial q_i,eq} \right) \dot{q}_i$$

And so:

$$T = \frac{1}{2} \sum \sum M_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{Q}^T M \dot{Q}$$

Where

$$M_{ij} = \sum \sum m \frac{\partial r}{\partial q_{i,eq}} \left| \frac{\partial r}{\partial q_{j,eq}} \right|$$

- Consider the potential energy, about a point of equilibrium (ie: a minimum in $U$) at which all the $q_i$ are chosen to be 0.

$$U(x) = U_0 + \sum \left. \frac{\partial U}{\partial q_i,eq} \right| q_i + \sum \sum \frac{1}{2} \left. \frac{d^2 U}{d x_i d x_j,eq} \right| q_i q_j + \cdots$$

$$U(x) = U_0 + \frac{1}{2} \sum \sum K_{ij} q_i q_j + \cdots$$

$$U(x) = U_0 + \frac{1}{2} q^T K q$$
The total energy is then
\[ E = U_0 + \frac{1}{2} \sum_i \sum_j M_{ij} \ddot{q}_i \ddot{q}_j + \frac{1}{2} \sum_i \sum_j K_{ij} q_i q_j \]
\[ \frac{dE}{dt} = \frac{1}{2} \sum_i \sum_j 2 \ddot{q}_i (M_{ij} \ddot{q}_j + K_{ij} q_j) = 0 \]
\[ \frac{dE}{dt} = \sum_i \sum_j \dot{q}_i (M_{ij} \ddot{q}_j + K_{ij} q_j) = 0 \]

[Non rigorous argument] – the equations of motion are then:
\[ \sum_i \sum_j M_{ij} \ddot{q}_j + \sum_i \sum_j K_{ij} q_j = 0 \]
\[ [M \ddot{q} + K q = 0] \]

If we seek normal modes of the form \( q(t) = Q e^{i \omega t} \), we get:
\[ (K - \omega^2 M)Q = 0 \]

Non-trivial solutions only exist if
\[ \text{det}(K - \omega^2 M) = 0 \]

This defines the \( \omega^2 \) normal mode frequencies.

In practice, the steps are:
  
  o Find the \( K \) and \( M \) matrices by writing them out in terms of the variables of the system, and comparing with
\[ T = \frac{1}{2} \dot{q}^T M \dot{q} \quad U = U_0 + \frac{1}{2} q^T K q \]

  Both matrices must be symmetric.
  
  o Use the determinant method above.
Dynamics – Elasticity

Introduction

- **Hooke’s Law** states that

\[
\frac{F}{A} = E \frac{\Delta l}{l}
\]

Where

- \( F \) is the *force* applied to a block of material over an area \( A \).
- \( \Delta l \) is the *extension* of the block *in the direction of* \( F \).
- \( l \) is the original, relaxed length of the block *in that direction*.
- \( E \) is the *Young’s Modulus* of the material.

- Furthermore, it states that

\[
\frac{\Delta w}{w} = -\sigma \frac{\Delta l}{l}
\]

Where \( \Delta w \) is the *length* of the block in *any direction* *perpendicular* to that of \( l \).

- For an *isotropic material*, \( E \) and \( \sigma \) are all we need to define the elastic properties of the material.

- Since these equations are all *linear*, the *principle of superposition* applies. If we have *several forces*, the *displacements* will be the *sum* of the displacements with the forces acting *individually*.

Uniform Strain – the Bulk Modulus

- Consider a *rectangular* block in a *pressure tank*, say, with *identical stress* \( p \) on every face.

- Consider one direction – the *change in length* \( \Delta l \) in that direction is given by

\[
\frac{\Delta l}{l} = \frac{-p}{E} + \sigma \frac{p}{E} + \sigma \frac{p}{E}
\]

\[
\frac{\Delta l}{l} = -1 - 2\sigma \frac{p}{E}
\]

The problem is *symmetrical*, so the value will be the same for all *directions*. 
- Now, consider the change in volume

\[
\frac{\Delta V}{V} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \frac{\Delta z}{z}
\]

We therefore have

\[
\frac{\Delta V}{V} = -3 \frac{1 - 2\sigma}{E} p
\]

- We can then define the bulk modulus

\[
K = \frac{E}{3(1 - 2\sigma)}
\]

Such that the change of volume as a result of the stress \( p \) is

\[
p = -K \frac{\Delta V}{V}
\]

Shear Strain – the Shear Modulus

- Consider a cube with face area \( A \) and with shear forces acting on it

If cut the cube along the diagonals \( A \) and \( B \), we find that

- There is a stretch normal to \( A \), of magnitude \( F\sqrt{2} \).
- There is a compression normal to \( B \), of magnitude \( F\sqrt{2} \).

And each of these diagonal faces has area \( A\sqrt{2} \).

- The lengthening of the diagonal \( d \) will therefore be equal to the lengthening of \( d \) in the following case:
From above, this is given by:

\[
\frac{\Delta d}{d} = \frac{1}{E} \frac{F \sqrt{2}}{A \sqrt{2}} + \frac{\sigma}{E} \frac{F \sqrt{2}}{A \sqrt{2}}
\]

\[
\frac{\Delta d}{d} = 1 + \frac{\sigma F}{E A}
\]

By symmetry, the other diagonal is shortened by the same amount.

- It is often useful to have this as a function of the twist angle:

From this diagram, it is (reasonably) clear that

\[
\delta = \Delta d \sqrt{2}
\]

\[
d = \ell \sqrt{2}
\]

Therefore

\[
\theta \approx \frac{\delta}{\ell} = \frac{\Delta d \sqrt{2}}{\ell} = 2 \frac{\Delta d}{d} = \frac{2(1 + \sigma) F}{E A}
\]

- We therefore define the shear modulus as

\[
\mu = \frac{E}{2(1 + \sigma)}
\]

Such that

\[
g = \mu \theta
\]

Where \( g \) is the shear stress \( = F/A \).
Formal Definitions

- **Stress**
  - Defined in terms of *force/unit area* transmitted across *planes* in the medium.
  - Requires a *tensor*. We define
    \[
    \tau_{ij}
    \]
    
    Force in the \( i \) direction
    On a plane perpendicular to the \( j \) axis

  - We can then show that the force on any arbitrary area element is
    \[
    F = \tau \, dS
    \]

  - The tensor must be *symmetric* – consider a *small cube* side \( dx \).
    Because the cube must be in *equilibrium*, the forces on it are as follows:

    \[
    S_{xx} \quad S_{xy} \quad S_{xz}
    \]

    \[
    S_{yx} \quad S_{yy} \quad S_{yz}
    \]

    \[
    S_{zx} \quad S_{zy} \quad S_{zz}
    \]

    The net *couple* on the cube is
    \[
    (S_{xy} - S_{yx}) \, dx
    \]

    But there must be no *torque* on the cube, or it’d spin! So
    \[
    S_{xy} = S_{yx}
    \]

  - The stress tensor is *diagonal* for suitable choices of axes.
  - The stress in a solid material is therefore described by a *tensor field*.

- **Strain**
  - When a material is put under *strain*, a point \((x,y,z)\) in it is moved to a point \((x+X,y+Y,z+Z)\).
  - The *derivatives* of these \(X, Y\) and \(Z\) contain information about the *strain*.
  - As we saw before, it’s worth considering *two kinds of strain*
    - For the *normal strains*, we define:
\[ e_{xx} = \frac{\partial X}{\partial x} \quad e_{yy} = \frac{\partial Y}{\partial y} \quad e_{zz} = \frac{\partial Z}{\partial z} \]

For example, if we consider stress **perpendicular** to the \( x \) **direction** in a **cube** initially of side \( \Delta x \), it’ll **increase** by \( e_{xx} \Delta x \):

- Now, for the **shear stresses**, consider

\[ e_{xy} = e_{yx} = \frac{1}{2} \left( \frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y} \right) \]

This ensures that if the block simply **rotates** (ie: \( \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y} \)), these strains are 0.

- So in general, we define

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial X_j}{\partial x_i} + \frac{\partial X_i}{\partial x_j} \right) \]

The \( i^{th} \) coordinate of a point in the material... \( \ldots \) will have changed by \( e_{ij} \Delta \), assuming that the \( j^{th} \) coordinate of that point from the origin is \( D \).

So, for example

\[ X = e_{xx} x + e_{xy} y + e_{xz} z \]

- The tensor is also **symmetric**, due to the \( e_{xy} = e_{yx} \) condition.

- If the strains are **non-homogenous**, we sit down and cry.

- **The relation between them**
Each component of $e$ is related to each component of $\tau$ – this gives, overall, a **fourth-rank tensor of elasticity** relating the two:

$$\tau_{ij} = C_{ijkl}e_{kl}$$

(Using the summation convention).

It looks like there are $9^2 = 81$ coefficients in $C$, and that 81 numbers are therefore required to define the elastic properties of a material! However, we note that since $S$ and $e$ are symmetry, we must be able to swap $ij$ and $kl$ in $C$ without changing a thing, so there can be at most 36 different coefficients.

If the material is isotropic, though, $C$ must be completely frame-independent. As such, we must be able to express it in terms of the tensor $\delta_{ij}$. There are only two ways of doing this that are also invariant under $i \leftrightarrow j$ and $l \leftrightarrow k$, and so

$$C_{ijkl} = \lambda(\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{jk} + \delta_{il}\delta_{jk})$$

So an isotropic material only requires two constants ($E$ and $\sigma$, for example). And we have

$$S_{ij} = \lambda e_{ik}\delta_{ij} + 2\mu e_{ij}$$

**Examples – Statics**

- **Thin tube in torsion**
  - Consider a thin tube being twisted an angle $\phi$

  ![Diagram of a thin tube in torsion](image)

  - We first note that

$$\theta = \frac{r\phi}{l}$$

- Next, consider a small square (dotted above) and its deformation as a result of the twist:
From the previous result:

\[
\frac{F}{\ell \Delta r} = \mu \theta
\]

\[
F = \mu \frac{r \phi}{L} \ell \Delta r
\]

- This force contributes a torque \( \Delta \tau \) to the rod

\[
\Delta \tau = r F = \mu \frac{r^2 \phi}{L} \ell \Delta r
\]

- Considering these bits around the whole rod, so that \( \ell \to 2\pi r \), we get

\[
\tau = 2\pi \mu \frac{r^3 \Delta r}{L} \phi
\]

- **Wire in torsion**

  For a wire, we simply integrate the above from \( r = 0 \) to the total radius, giving

\[
\tau = \mu \frac{\pi r^4}{2L} \phi
\]

- **Can under pressure**

  - Consider a can of thickness \( t \) with closed ends with an internal pressure \( p \).

  - Let the tangential stress in the walls be \( \tau_\theta \), and consider half the can

    \[
    \begin{array}{c}
    \tau_\theta \\
    p \\
    \tau_\theta \\
    \hline
    r \\
    \hline
    \end{array}
    \]

    The forces (= stress \( \times \) area) must balance, so

    \[
    \tau_\theta \times 2t = p \times 2r
    \]

    \[
    \tau_\theta = \frac{pr}{t}
    \]

  - Let the axial stress in the walls be \( \sigma_z \), and consider one of the ends. By the same logic as above
\[ \tau_z \times 2\pi rt = p \times \pi r^2 \]
\[ \tau_z = \frac{pr}{2t} \]

- **Bent beam**
  - Consider a beam of length \( L \), held in a bent position.
  - We only consider longitudinal strains (valid for small deflections and thin beams).
  - Clearly, the bits at the top of the beam will be stretched, while those at the bottom will be compressed. Somewhere in between, there’ll be a neutral surface – neither stretched nor compressed.
  - Consider a small segment length \( \ell \) of the bent beam:

\[ \Delta \ell = \text{Strain} = \frac{y}{R} \]

- The amount of stretching and compression at any point is proportional to the distance from the neutral surface, \( y \). The constant of proportionality is \( \Delta \ell / R \). As such

\[ \Delta F = \frac{E}{R} y \Delta A \]

- The total torque produced about the neutral line is given by

\[ \tau = \int_{\text{section}} y \, dF \]
\[ = \frac{E}{R} \int_{\text{section}} y^2 \, dA \]
\[ B = \frac{EI}{R} \]
Now, consider a beam loaded with weights given by $W(x)$, where $W$ is the force per unit length. Consider the statics of a small segment of the beam:

$$S + dS$$

Now, for small deflections

$$y'' = 1/R$$

As such, we can conclude

$$EIy''' = W(x)$$

Boundary conditions for various cases are as follows

- At a free end, $S$ and $B$ are clearly 0, and so $y'' = y''' = 0$.
- At a cantilevered end, $y$ and $y'$ are given (usually 0).

Finding $y$ is then simply a question of solving that differential equation. However, there are a few tricky points

- All forces must be considered when writing down $W(x)$, including reactions at contacts. Most often, $W$ will be a series of $\delta$-functions.
- Sign conventions:
• **Downwards** $W \rightarrow$ **positive**.
• The resulting $y$ obtained is **downwards** $\rightarrow$ **positive**, because the way the radius of curvature is specified.
• However, be **very** careful – sometimes, the convention appears to be reversed because the bar curves downwards, and so $-1/R = y''$.

- Don’t worry too much about boundary conditions for $y'''$ – just integrate $\delta$-functions from 0 to $L$ (for a free end, this is fully justified). Remember that there’ll often be a $\delta$-functions at the **very end** of the range, which might help satisfy the boundary conditions.

- From then on, **boundaries** are just provided. Just also remember to make the $y''$, $y'$ and $y$ **continuous**.

- The **couple** provided by a **cantilever** can simply be worked out by evaluating $B = EIy''$ at that point.

- It is sometimes easier to simply **write down** $y''$, the **bending moment** from **physical considerations**.

  - The **Euler Strut** is a beam **buckled** between **two walls**:

    ![Euler Strut Diagram]

    If we take $y$ **upwards**, then the **bending moment** on any point is
    
    $B = -Fy$
    
    $y'' = -\frac{F}{EI}y$
    
    $y = A\sin\left[\sqrt{\frac{F}{EI}}x\right]$

    Applying the boundary condition that $y = 0$ at $x = L$:
    
    $F = \frac{\pi^2EI}{L^2}$

    This is **independent of displacement** (but only while $y'' = 1/R$ holds).

  - The **Reciprocity Theorem** states that
    
    “**The deflection at Q due to a load at P is the same as the deflection at P due to the same load at Q**”

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To prove, say $P_{PQ}$ means “the deflection at $P$ due to the load at $Q$”. Consider loading first $P$ and then $Q$. The energy stored is

$$E = F \left( \frac{P_{PP}}{2} + \frac{P_{QQ}}{2} + P_{PQ} \right)$$

The same result must be applied the other way round, so

$$P_{PQ} = P_{QP}$$

**Dynamics of Rigid Bodies**

- Consider a **small volume** $V$ of the material. It will have both external forces acting on it (eg: gravity) and internal forces (eg: elastic stresses).

$$F_{\text{ext}} + F_{\text{int}} = \int \rho \dd \mathbf{r} \, dV$$

- Every small particle in the volume experiences the external force, though, so $F_{\text{ext}}$ is given by a volume integral.

$$F_{\text{int}} = \int \left( -f_{\text{ext}} + \rho \dd \mathbf{r} \right) \, dV$$

$$F_{\text{int}} = \int \mathbf{f} \, dV$$

On the other hand, only the particles at the edge of the volume experience the elastic force from surrounding media, and so $F_{\text{int}}$ is given by an area integral

$$\int_{\Delta} f_{\text{int}} \, dA = \int \mathbf{f} \, dV$$

- We have, however, defined that the force in the $x$-direction, say, is

$$dF_x = (S_{xx} \mathbf{i} + S_{xy} \mathbf{j} + S_{xz} \mathbf{k}) \cdot d\mathbf{A}$$

And so, taking only the $x$ component of the integral above

$$\int_{\Delta} (S_{xx} \mathbf{i} + S_{xy} \mathbf{j} + S_{xz} \mathbf{k}) \cdot d\mathbf{A} = \int \mathbf{f}_x \, dV$$

- Using the Divergence Theorem on the LHS

$$\int \left( \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{xz}}{\partial z} \right) \, dV = \int \mathbf{f}_x \, dV$$

Removing the volume integrals (because this is true for any volume):

$$\mathbf{f}_x = \partial S_{ij}/\partial x_j$$

(Using the summation convention).

- Now, using $\tau_{ij} = \lambda \epsilon_{ij} + 2\mu \delta_{ij}$ (isotropic material), we obtain

$$\mathbf{f} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}$$

Where $\mathbf{u}$ is the internal displacement in the solid.