Waves

The Wave Equation

- Let a wave propagate in the x direction, such that its shape at $t = 0$ is given by $f(x)$ – ie: $\psi(x,0) = f(x)$.
- If the wave is travelling **forwards** with speed c, then we know that the following holds:

$$\psi(x,t) = \psi(x - ct, 0) = f(x - ct) = f(u)$$

Because $x - ct$ is the position the current point of the wave was at when t was equal to 0. Similarly, for a wave travelling in the *negative* x direction:

$$\psi(x,t) = \psi(x + ct, 0) = f(x + ct) = f(u)$$

- We can then work out partial derivatives:

$$\frac{\partial \psi}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u} \Rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}f}{\mathrm{d}u}\right)\frac{\partial u}{\partial x} = \frac{\mathrm{d}^2 f}{\mathrm{d}u^2}}$$

$$\frac{\partial \psi}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial t} = -c\frac{\mathrm{d}f}{\mathrm{d}u} \Rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} = -c\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}f}{\mathrm{d}u}\right)\frac{\partial u}{\partial t} = c^2\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}}$$

Therefore,

$$\boxed{\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}}}$$

This is the **wave equation** in **1-dimension**.

- The most general solution to the wave equation is

$$\psi(x,t) = f(x - ct) + g(x + ct)$$

ie: two waves travelling in opposite directions.

- In two and three dimensions, this becomes

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

Properties of waves

- Consider a "typical" wave of the form

$$\boxed{\psi(x,t) = A\cos(\omega t + kx)}$$

Now:

  - The wave must repeat every wavelength – so:

$$A\cos(\omega t + kx) = A\cos(\omega t + k[x+\lambda])$$
$$\cos(\omega t + kx) = \cos(\omega t + kx + k\lambda)$$

Therefore, $k\lambda = 2\pi$, and

$$\boxed{k = \frac{2\pi}{\lambda}}$$

Where k is called the **wavenumber**.

- o The wave must repeat every period – so:

$$A\cos(\omega t + kx) = A\cos(\omega[t+T] + kx)$$
$$\cos(\omega t + kx) = \cos(\omega t + kx + \omega T)$$

So $\omega T = 2\pi$, and

$$\boxed{\omega = \frac{2\pi}{T}}$$

- o If the speed of the wave is c [depending on the characteristics of the medium], then

$$\boxed{\lambda = Tc}$$

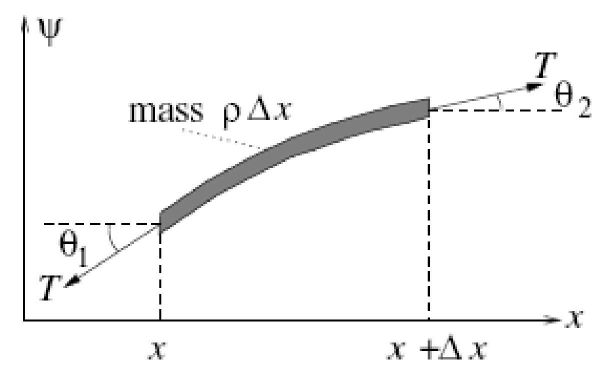
This allows us work out that

$$\boxed{\omega = ck = \frac{2\pi c}{\lambda}}$$

- o We could also have gone backwards and, noting that $\psi(x,t) = \psi(x-ct,0)$, have derived all these properties.

- The speed of a particular particle in the wave is given by $\frac{\partial \psi}{\partial t}$.

Waves on strings



- By Newton's Second law,

$$T\sin\theta_2 - T\sin\theta_1 = (\rho\Delta x)\frac{\partial^2\psi}{\partial t^2}$$

- For small angles,

$$\sin\theta_1 \approx \tan\theta_1 = \left(\frac{\partial\psi}{\partial x}\right)_x \qquad \text{and} \qquad \sin\theta_2 \approx \tan\theta_2 = \left(\frac{\partial\psi}{\partial x}\right)_{x+\Delta x}$$

- So, the statement above becomes

$$\left(\frac{\partial \psi}{\partial x}\right)_x + \left(\frac{\partial \psi}{\partial x}\right)_{x+\Delta x} = \frac{\rho}{T}\frac{\partial^2 \psi}{\partial t^2}\Delta x$$

$$\frac{\Delta\left(\frac{\partial \psi}{\partial t}\right)}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 \psi}{\partial t^2}$$

- In the limit, as $\Delta x \to 0$,

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\rho}{T}\frac{\partial^2 \psi}{\partial t^2}$$

Which is in fact the wave equation, with

$$\boxed{c^2 = \frac{T}{\rho}}$$

- In problems, always remember to convert ρ from the given value for an unstretched string to a value for a stretched string.
- The assumptions here are that:
  - The tension, T, remains constant.
  - $\sin\theta \approx \tan\theta$

This derivation therefore only applies to *small* displacements of the string.

- To find the energy and power in a wave on a stretched string, we note that it is make up of KE and PE.
- The KE in a length Δx of the string is given by

$$\tfrac{1}{2}\rho\Delta x\left(\frac{\partial \psi}{\partial t}\right)^2$$

Thus, the KE **per unit length** (the KE density) is given by:

$$\tfrac{1}{2}\rho\left(\frac{\partial \psi}{\partial t}\right)^2$$

- The length of the small bit of string is given by

$$\sqrt{\Delta x^2 + \Delta\psi^2}$$

Bearing in mind that $\Delta\psi / \Delta x \approx \partial\psi / \partial x$, we can write

$$\begin{aligned}\sqrt{\Delta x^2 + \Delta\psi^2} &= \Delta x\sqrt{1 + \left(\frac{\partial \psi}{\partial x}\right)^2} \\ &\approx \Delta x\left[1 + \tfrac{1}{2}\left(\tfrac{\partial \psi}{\partial x}\right)^2\right] \\ &= \Delta x + \tfrac{1}{2}\Delta x\left(\tfrac{\partial \psi}{\partial x}\right)^2\end{aligned}$$

The extension of this small piece of string is therefore

$$\tfrac{1}{2}\Delta x\left(\tfrac{\partial \psi}{\partial x}\right)^2$$

And the PE is therefore

$$\Delta\text{PE} = \tfrac{1}{2}T\Delta x\left(\tfrac{\partial\psi}{\partial x}\right)^2$$

[**Note:** there is no factor of T^2 because we assume T is constant – we therefore just multiply tension by extension; no need to integrate].

The **PE per unit length** (PE density) is therefore simply

$$\tfrac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2$$

- For a harmonic wave of the form $\psi = A\cos(\omega t - kx)$, the total energy density is

$$\rho\omega^2 A^2 \sin^2(\omega t - kx)$$

And so the **time average** of energy density [integrate over a period and divide by the period] is

$$\tfrac{1}{2}\rho\omega^2 A^2$$

Each element of length Δx on the string can therefore be considered as a harmonic oscillator with energy $\frac{1}{2}\rho\omega^2 A^2\Delta x$, and in a time Δt, an extra length of sting $c\Delta t$ is set oscillating. Thus, the average power input is given by

$$\tfrac{1}{2}\rho\omega^2 A^2 c$$

Waves in 2-Dimensions

- **Plane waves** are those for which the **amplitude** along the **wavefronts** (lines of constant phase) is **constant**.
- Plane waves in more than one dimension have the formula:

$$\psi(\boldsymbol{x}, t) = A\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})$$

Where $\boldsymbol{x}$ is the position vector of a point in the plane, and $\boldsymbol{k}$ is a vector in the direction of the propagation of the wave.

This can be demonstrated by using the transformation $x' = x\cos\theta + y\sin\theta$, where the primed frame is the one along whose x-axis the wave is travelling. In that frame, $k = |\boldsymbol{k}|$.

Standing Waves

- When considering standing waves from first principle, the trigonometric identity to use is

$$\cos B + \cos C = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

- By the **principle of superposition**, any standing wave will also be a solution of the wave quation.
- **Nodes** in a wave indicates that there is **no net transfer of energy**.
- In a **travelling wave**, all points undergo SHM with a **different phase**. In a **standing wave**, all points undergo SHM with the **same phase**.
- Boundary conditions
    - For pipes
        - At an **open end**, $\left(\frac{\partial \psi}{\partial x}\right)_{x=end} = 0$ at all times – this is because the pressure of the air molecules at the end of the pipe must always be the same as that of the atmosphere – therefore, we have a pressure node! [In fact, this occurs slightly outside the end of the pipe].
        - At a **closed end**, $\psi = 0$ at all times – the air has nothing else to squash after itself, so it must just stand still.
    - For strings and drums
        - At a point of attachment, $\psi = 0$ at all times.
- If a string is "touched" at a given point, then there is a node there. Intuition must then be used to determine what frequencies can then exist.

  We need sub-integer multiples of the length of the smaller side of the string. We can just have different wavelengths on either side of the touch, because the tension on both sides of the node must be the same.
- When attempting standing waves, try something of the form

$$\psi(x,y,z,t) = X(x)Y(y)Z(z)\cos\omega t$$

  Where $X(x) = A_x \cos k_x x + B_x \sin k_x x$ and Y and Z have analogous forms, such that

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

  Note that for waves which are 0 at $x = 0$, $A_x = 0$, and similarly for y and z.

- When drawing standing waves in 2D, do squares with “+ve” or “—ve” written in. The sign of the function changes at nodes.