8.033 Relativity ~ Additional Notes

Aberration and the Doppler Effect

- If a particle is moving with speed u at an angle θ to the horizontal, then, in a frame moving horizontally at a speed v, the horizontal and vertical velocities will be

$$u_x' = \frac{u\cos\theta - v}{1 - \frac{u_x v}{c^2}} \qquad u_y' = \frac{u\sin\theta}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

And so

$$\boxed{\tan\theta' = \frac{u\sin\theta}{\gamma\left(u\cos\theta - v\right)}}$$

- When using the **aberration** and **Doppler** formulae, the **angle** is the one between the **direction of motion of the photon** and the **direction in which the observer is moving**.
- Relations for a photon $E = pc = \hbar\omega = hc / \lambda$

Extra Dynamics Stuff

- To find a threshold energy, evaluate $E^2 - p^2c^2$ in the ZMF *after* the collision (in which p is 0)
- Equate it to the invariant before the collision in the lab frame.

4-Vectors, Formally

- **4-vectors** are vectors that **transform** like **(c dt, dx, dy, dz)**.
- In deriving them, we use the fact that τ is **invariant** and

$$\mathrm{d}\tau = \sqrt{\mathrm{d}t^2 - \mathrm{d}\boldsymbol{r}^2} = \mathrm{d}t\sqrt{1 - \dot{\boldsymbol{r}}^2} = \frac{\mathrm{d}t}{\gamma}$$

- We also note that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - v^2\right)^{-1/2} = -\frac{\cancel{1}}{2}\left(1 - v^2\right)^{-3/2} \cdot \left(-\cancel{2}v\frac{\mathrm{d}v}{\mathrm{d}t}\right) = \gamma^3 v\dot{v}$$

- Examples are
  - **Velocity 4-vector** – obtained by dividing every component above by the **proper time** $(\mathrm{d}\tau)$. Since $\mathrm{d}\tau$ is invariant, the result is a 4-vector

$$\boldsymbol{U} = \frac{1}{\mathrm{d}\tau}\left(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\right) = \gamma\frac{1}{\mathrm{d}t}\left(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\right) = \gamma\left(1, \boldsymbol{u}\right)$$

    Note – the γ refers to the $\boldsymbol{u}$, because the dx, etc… are taken **in the frame of the moving object**.
  - **Energy-momentum 4-vector** obtained by multiplying $\boldsymbol{U}$ by the **invariant** m.

$$\boldsymbol{P} = m\boldsymbol{U} = \gamma\left(m, m\boldsymbol{u}\right) = \left(E, \boldsymbol{p}\right)$$

  - **Acceleration 4-vector** – obtained by taking the **derivative** of the **velocity 4-vector** with **respect to $\boldsymbol{\tau}$**.

$$\begin{aligned}\boldsymbol{A} = \frac{\mathrm{d}V}{\mathrm{d}\tau} &= \gamma\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}(\gamma\boldsymbol{u})}{\mathrm{d}t}\right) \\ &= \gamma\left(\gamma^3 u\dot{u}, \gamma^3 u\dot{u}\boldsymbol{u} + \gamma\boldsymbol{a}\right)\end{aligned}$$

    If $\boldsymbol{u}$ points along the $\boldsymbol{x}$-direction

$$\begin{aligned}\boldsymbol{A} &= \gamma\left(\gamma^3 u_x a_x, \gamma^3 u_x^2 a_x + \gamma a_x, \gamma a_y, \gamma a_z\right) \\ &= \gamma\left(\gamma^3 u_x a_x, \gamma\left(\gamma^2 u_x^2 + 1\right) a_x, \gamma a_y, \gamma a_z\right) \\ &= \left(\gamma^4 u_x a_x, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z\right)\end{aligned}$$

  - **Force 4-vector** – obtained by taking the **derivative** of the **momentum 4-vector** with **respect to $\boldsymbol{\tau}$**.

$$\boldsymbol{\mathcal{F}} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \gamma\left(\frac{\mathrm{d}E}{\mathrm{d}t}, \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right) = \gamma\left(\frac{\mathrm{d}E}{\mathrm{d}t}, \boldsymbol{F}\right)$$

    If the **mass** is **constant**, we have $\boldsymbol{\mathcal{F}} = m(\mathrm{d}\boldsymbol{U} / \mathrm{d}t) = m\boldsymbol{a}$, and we can write

$$\boldsymbol{\mathcal{F}} = m\gamma\left(\gamma^3 u\dot{u}, \gamma^3 u\dot{u}\boldsymbol{u} + \gamma\boldsymbol{a}\right)$$

    Comparing these two expressions

$$\boldsymbol{F} = m\gamma^3 u\dot{u}\boldsymbol{u} + m\gamma\boldsymbol{a}$$

    If $\boldsymbol{u}$ is in the $\boldsymbol{x}$-direction, we can use the result above

$$\boxed{\boldsymbol{F} = m\left(\gamma^3 a_x, \gamma a_y, \gamma a_z\right)}$$

    A very useful result indeed.

- **Current density 4-vector** – obtained by **multiplying** the **velocity 4-vector** by the **proper charge density** $\rho_0 = Q / V_0$, where V_0 is the **rest volume** of the charge.

$$\boldsymbol{J} = \rho_0 \gamma \left(1, \boldsymbol{u}\right)$$

  We note that
    - The **first component**, is the **effective charge density**, because the **effective volume** is V_0 / γ.
    - The **other components** are the **effective current densities**.

- **Velocity transformation** can easily be derived by considering **two particles** moving **away from each other** in **one frame**, and taking the **inner product**.
- **Energy momentum calculations** can easily be done with **4-vectors**. Particularly
  - For a **single particle**, we **recover** $E^2 - p^2 = m^2$
  - For a **group of particles**, we get

$$\left(\sum E\right)^2 - \left(\sum \boldsymbol{p}\right)^2 = \text{Sum of energies in ZMF}$$

- We can derive the **transformation of forces** from the above
  - In the **rest frame** of a particle

$$\boldsymbol{\mathcal{F}} = \gamma \left(\frac{\mathrm{d}E'}{\mathrm{d}t}, \boldsymbol{F}\right) = \left(0, F_x, F_y, F_z\right)$$

  - As such, in a **new frame**

$$\boldsymbol{\mathcal{F}}' = \left(\gamma v F_x, \gamma F_x, F_y, F_z\right)$$

  But using the definition of the force 4-vector

$$\boldsymbol{\mathcal{F}}' = \gamma \left(\frac{\mathrm{d}E'}{\mathrm{d}t}, F_x', F_y', F_z'\right)$$

  And so

$$\left(\frac{\mathrm{d}E'}{\mathrm{d}t}, F_x', F_y', F_z'\right) = \left(\frac{\mathrm{d}x}{\mathrm{d}t} F_x, F_x, \frac{F_y}{\gamma}, \frac{F_z}{\gamma}\right)$$

$$\boxed{\left(\mathrm{d}E', F_x', F_y', F_z'\right) = \left(F_x \, \mathrm{d}x, F_x, \frac{F_y}{\gamma}, \frac{F_z}{\gamma}\right)}$$

  - The last three components give us the **law of transformation of force**, and the **first** gives us the **work-energy Theorem**.
- We can also derive the **transformation of accelerations** from the above

- In the **rest frame** of a particle

$$\boldsymbol{A} = (0, a_x, a_y, a_z)$$

- As such, in a **new frame**

$$\boldsymbol{A}' = \left(\gamma u a_x, \gamma a_x, a_y, a_z\right)$$

But we also know, using the definition of $\boldsymbol{A}$ if $\boldsymbol{u}$ points in the x-direction, that

$$\boldsymbol{A}' = \left(\gamma^4 u_x a_x', \gamma^4 a_x', \gamma^2 a_y', \gamma^2 a_z'\right)$$

And so

$$\boxed{\left(a_x', a_x', a_y', a_z'\right) = \left(\frac{a_x}{\gamma^3}, \frac{a_x}{\gamma^3}, \frac{a_y}{\gamma^2}, \frac{a_z}{\gamma^2}\right)}$$

GR & Metrics

- The **central principle** of **GR** is that "***mass*** *tells* ***spacetime*** *how to* ***curve***, ***spacetime*** *tells* ***mass*** *how to* ***move***".
- The **Einstein Field Equations** allow us to determine **how** spacetime **curves** given the masses present. The result is a **metric**, which is an expression for the **spacetime interval** between **two events**.
- **Metrics** are formulated such that the **wristwatch time** measured along a **curve** $\boldsymbol{r}(t)$ through spacetime is given by

$$\Delta\tau = \int \mathrm{d}\tau$$

- The **principle of extremal aging** states that the **path** of a **massive object** from A to B is the **geodesic** between these two points – in other words, the path that **extremises** $\Delta\tau$ above.
- A **photon** will move along a **null geodesic**, with $\Delta\tau = 0$.
- The simplest metric studied so far is the **Minkowski Metric**

$$\mathrm{d}\tau = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

This can be expressed in **polar coordinates** as

$$\mathrm{d}\tau = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$$

Where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \mathrm{d}\phi^2 \sin^2\theta$$

We have

- **General relativity** allows us to express things in **any frame** using the **general covariance** to convert to **any** such frame.

The Newtonian Metric

- **Matter** affects the **gravitational field** as $\nabla^2\phi = 4\pi G\rho$ and the **gravitational field** affects **matter** as $F = -m\nabla\phi$.
- The **equivalence principle** comes in two forms
  - **Weak equivalence principle** – no local experiment can distinguish between a uniform gravitational field ***g*** and a frame accelerating with ***a*** = ***g***.
  - **Strong equivalence principle** – the laws of physics take on their special-relativistic forms in any **locally inertial frame**.

  These arise from the fact that **inertial mass = gravitational mass**, and that the **gravitational acceleration** is therefore **independent of mass**.
- This implies that gravity is a **purely geometric object**. If we know the **worldline** of an object moving under the influence of **gravity** (ie: **velocity 4-vector**), its **future trajectory** does **not** depend on its **mass** or **composition**.
- This **implies** that the **lower** we are in a **gravitational potential**, the **slower** our **clocks will run**.

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 + \frac{\phi}{c^2}$$

  Where τ is the **measured (wristwatch) time** and t is the **far-away time**, measured **far away from any potential**. This implies that we must **modify our metric**

$$\boxed{\mathrm{d}\tau^2 = \left(1 + 2\phi\right)\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}$$

  [Where we have used a Taylor Expansion and used $c = 1$]. This is the **Newtonian Metric**.
- To derive the relation above

- Consider **two pulses** send τ_ϕ apart at a point with potential ϕ. They are received at a time τ_0 apart at a point with potential 0.
- This is **equivalent** to the pulses being send from the **top** to the **bottom** of an elevator in **free fall g** and **height ϕ / g**.
- The pulses have a distance $\tau_\phi c$ between them, and take a time ϕ / gc to travel to the bottom of the elevator.
- By that time, the bottom of the elevator is moving at a speed $(\phi / gc) \times g = \phi / c$, and so the relative velocity between the bottom of the elevator and the photons is $(\phi / c) + c$.
- As such, the separation in time between the two pulses arriving is

$$\tau_0 = \frac{\tau_\phi c}{(\phi / c) + c} \Rightarrow \boxed{\frac{\tau_\phi}{\tau_0} = \frac{\phi}{c^2} + 1}$$

- Using the metric above, we can **re-derive Newton's Laws** for the geodesics.

Cosmology

- The most general metric that is isotropic and homogeneous is the FRW metric:

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2 \right)$$

- Lines in space with constant r, θ, ϕ are **geodesics**, and objects on such lines are **commoving** at that **commoving position**. **Galaxies** are (approximately) commoving.
- The distance between commoving objects *increases* with time, proportionally to *a(t)*. Space **streches**. Furthermore, *a* can often be used as a convenient variable instead of *t*.
- If $k = 0$, this is just **expanding flat space**
- Light emitted at a time a_1 and observed at a time a_2 will have red-shifted due to space stretching. The **red-shift** (***z***) is defined as

$$1 + z = \frac{\lambda_2}{\lambda_1} = \frac{a_{\text{now}}}{a_1}$$

When we say something happened "at red-shift z", it means that it was emitted at a time such that the light emitted will now be observed with a red-shift z. $z = 0$ is now, $z = \infty$ is the big bang, and $z = -1$ is the distant future.

- It turns out that the FRW metric is only a solution of the Einstein Field Equations if it obeys the **Friedmann Equation**

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}}$$

Where H is the **Hubble Parameter**, defined as above, and ρ is the density of stuff in space. The "stuff" is photons, ordinary matter and dark energy. We usually also define $\rho_k \equiv -3kc^2 / 8\pi G a^2$, so that we can write

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_\gamma + \rho_\Lambda + \rho_k\right)$$

- The **cosmological parameter** Ω for "stuff" in the universe is given by **the density of the stuff** divided by the **critical density**, which is defined as

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$$

As such $\Omega_k = 1 - \Omega_{\text{tot,matter}}$. So for $\Omega_{\text{tot,matter}} > 1$, space is finite (like a hypersphere), and for $\Omega_{\text{tot,matter}} \leq 1$, space is infinite.

- Different stuff, however, dilutes differently in space
  - $\rho_\gamma \propto a^{-4} \propto (1+z)^4$ [photons]
  - $\rho_m \propto a^{-3} \propto (1+z)^3$ [ordinary matter]
  - $\rho_k \propto a^{-2} \propto (1+z)^2$ [special curvature; the k-term in the Friedmann eq.]
  - $\rho_\Lambda = \text{constant}$ [vacuum energy; the cosmological constant]

A few points

  - We can also find dependence on t by feeding into the Friedmann Equation, using the fact that $H^2 = (\dot{a}/a)^2 \propto \rho$ and solving.
  - We can find exact expressions for all of these in terms of **current** densities by noting that
    - Currently, $z = 0$.
    - Currently, $a = 1$ (as we scale it).

  And so, for example, $\rho_\gamma = \rho_{\gamma,\text{now}} a^{-4} = \rho_{\gamma,\text{now}}(1+z)^4$

We can therefore re-write the Friedmann Equation in terms of t, a or z instead of ρ.

- In many cases, it also helps to just write everything in terms of cosmological factors Ω instead of densities ρ.
- The "Hubble constant" (dimensionless Hubble parameter) is defined as

$$h = \frac{H_0}{100\,\mathrm{kg\,s^{-1}\,Mpc^{-1}}} \approx H_0 \times 9.78\ \mathrm{Gyr}$$

The Schwarzschild Metric & Black Holes

- Units
  - **Mass in geometrical (length) units** is given by $M = (G / c^2) M_{\mathrm{kg}}$.
  - **Length in geometrical (time) units** is given by $D = D_{\mathrm{m}} / c$
- Timelike Schwarzschild Metric in geometric units

$$\boxed{\mathrm{d}\tau^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2}$$

With $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

- Coordinate systems:
  - **Schwarzschild Bookeeper**
    - ***r*** at a shell is the **reduced circumference** ($Circumference\ /\ 2\pi$).
    - ***t*** is the **far-away time**. It can be determined for an event at the shell either by (1) sending light signals out and timing their difference far away *or* (2) keeping a clock at the shell that runs at a different, corrected rate and is synchronised.
  - **Free-float**
    - Coordinates as "seen" by a freely falling observer.
    - No force felt in such a frame.
  - **Shell**
    - Coordinates as "seen" by an observer seated on a shell around the black hole.

    - Only possible **outside the Horizon**. Inside the horizon, the **character** of **time** and **space** are reversed, and just as time **irresistibly moves forward** outside the radius, **so must space** inside the radius – a shell cannot stay stationary.
    - Because of the departure from natural motion entailed by sitting on a shell, we need a **force** to keep us there. We interpret this as the "fictitious" **force of gravity**.
    - Substituting values for $\mathrm{d}t_{\text{shell}}$ and $\mathrm{d}r_{\text{shell}}$ into the metric reveals that spacetime **looks** flat in the shell frame. Most importantly, the **speed of light** there is measured to be **1** (most easily proved by substituting $\mathrm{d}r_{\text{shell}}{}^2 + r^2\mathrm{d}\phi = \mathrm{d}x_{\text{shell}}$ and noting that $\tau = 0$ for light).
    - Information can be exchanged between the **free-fall frame** and the **shell frame** using **special relativity** (see below).
  - **Conversions**
    - Successively set $\mathrm{d}r = 0$ and $\mathrm{d}t = 0$ in the metric to find the **proper time** and **proper length** at a **shell**. We obtain

$$\boxed{\mathrm{d}t_{\text{shell}} = \gamma_r^{-1}\,\mathrm{d}t} \qquad \boxed{\mathrm{d}r_{\text{shell}} = \gamma_r\,\mathrm{d}r}$$

With $\gamma_r \equiv \left(1-\beta_r^2\right)^{-1/2}$ and $\beta_r \equiv \sqrt{2M / r}$.

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- **Conserved Quantities**
  - Consider a particle travelling into a black hole. Its wristwatch time must be extremized (we assume an orbit at a constant θ):

$$\tau = \int \mathrm{d}t\sqrt{\left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2} = \int F\,\mathrm{d}t$$

The Euler-Lagrange Equation for ϕ gives

$$\frac{\partial f}{\partial\dot{\phi}} = \text{constant} \Rightarrow -r^2\frac{\mathrm{d}\phi / \mathrm{d}t}{F} = \text{constant} \Rightarrow -r^2\frac{\mathrm{d}\phi / \mathrm{d}t}{\mathrm{d}\tau / \mathrm{d}t} = \text{constant}$$

$$\boxed{\tilde{L} = \frac{L}{m} = r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}}$$

And for r, we note that the functional does not include t explicitly, and we assume that the orbit is completely radial ($\mathrm{d}\phi = 0$) so:

$$F - \dot{r}\frac{\partial f}{\partial \dot{r}} = \text{constant}$$

$$F + \frac{1}{F}\left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \text{constant}$$

$$F + \frac{1}{F}\left\{\left(1 - \frac{2M}{r}\right) - F^2\right\} = \text{constant}$$

$$\boxed{\tilde{E} = \frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}}$$

- These two quantities are therefore **conserved**.

- **The Energy**
  - The energy is the **total unified energy of the system**. (It can most easily calculated by finding its **effect on a mass at a great distance** – the resulting "apparent mass" is equal to the energy).
  - We can calculate the energy E in two important cases. Since E is constant, this value will remain for the whole orbit.
    - When the particle is launched from rest at infinity, $\mathrm{d}\tau = \mathrm{d}t$ and $r = \infty$, and so

$$\boxed{\tilde{E} = 1}$$

    - When the particle is launched from a radius r, at a speed with gamma factor γ_v, we first note that $\mathrm{d}t_{\text{shell}} = \gamma_v \mathrm{d}\tau$ (using special relativity). We then note that $\mathrm{d}t = \gamma_r \mathrm{d}t_{\text{shell}}$. As such

$$\boxed{\tilde{E} = \gamma_v\sqrt{1 - \frac{2M}{r}}}$$

- **The Angular Momentum** – can be calculated using very similar methods to the above. The most general case is a launch at speed v_0, angle θ_0 to the outwards radial from a shell r. In that case

$$v_{\text{tangential}} = v_0 \sin\theta_0 = r\tfrac{\mathrm{d}\phi}{\mathrm{d}t_{\text{shell}}}$$

We multiply by $\mathrm{d}t_{\text{shell}} / \mathrm{d}\tau = \gamma_{v_0}$, and get

$$r\tfrac{\mathrm{d}\phi}{\mathrm{d}\tau} = \gamma_{v_0} v_0 \sin\theta_0$$

From which we get

$$\boxed{L = r\gamma_{v_0} v_0 \sin\theta_0}$$

- Miscellaneous points
  - The red-shift of light from a shell can be deduced using conversions between shell and far-away coordinates, *either* by considering the time between two peaks in a wave, *or* the distance between such peaks.

Motion Around & In a Black Hole

- **Radial Plunge**
  - Consider a particle with energy $\tilde{E}$ radially plunging into a black hole. We have that

$$\tilde{E} = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \Rightarrow \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = \frac{1}{\tilde{E}^2}\left(1 - \frac{2M}{r}\right)^2$$

And from the Schwarzchild metric, we have that (for a radial plunge)

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = \left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2$$

And so combining the two, we get

$$\boxed{\frac{\mathrm{d}r}{\mathrm{d}t} = -\left(1 - \frac{2M}{r}\right)\sqrt{1 + \frac{1}{\tilde{E}^2}\left(\frac{2M}{r} - 1\right)}}$$

We take the **negative root** because this describes a **plunge** – radius **decreases** with time.

  - The **SHELL OBSERVER**, however, measures something different

$$\boxed{\frac{\mathrm{d}r_{\text{shell}}}{\mathrm{d}t_{\text{shell}}} = \frac{\gamma_r \mathrm{d}r}{\gamma_r^{-1}\mathrm{d}t} = -\sqrt{1 + \frac{1}{\tilde{E}^2}\left(\frac{2M}{r} - 1\right)}}$$

The **energy measured at the shell** (E_{shell}) can be calculated using special relativity, which the shell observer can use:

$$\boxed{\tilde{E}_{\text{shell}} = \gamma_v = \tilde{E}\left(1 - \frac{2M}{r}\right)^{-1/2}}$$

This is a **general formula**, which applies **even for photons**.

- To find the point of view of the **PLUNGING OBSERVER** (wristwatch time $\tau = \mathrm{d}t_{\text{rain}}$), we note that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \tilde{E}\left(1 - \frac{2M}{r}\right)^{-1}$$

And so

$$\boxed{\frac{\mathrm{d}r}{\mathrm{d}t_{\text{rain}}} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = -\tilde{E}\sqrt{1 + \frac{1}{\tilde{E}^2}\left(\frac{2M}{r} - 1\right)}}$$

[The reduced circumference takes meaning even inside the radius, since the free-faller can simply measure a piece of circumference and deduce r].

- Salient points are:
  - The **faraway observer** never sees the object cross the horizon. It simply comes to a slow stop before it, and **red-shifts out of sight**.
  - The **shell observer** at the radius *would* see the object whizzing by at the **speed of light**. However, shells cannot exist at or past the horizon, so this is purely academic.
  - For the **falling observer**, we can find the time from **horizon** to **crunch** by integrating $\mathrm{d}\tau$ given by the last equation above. We get $\tau = 4M/3$ from an observer from infinity, or $\tau = \pi M$ from an observer *just* from the horizon.
  - The energy $\boldsymbol{E}_{\text{shell}}$ is a **local quantity** and **not a constant of the motion**.
- **The Gullstrand-Painlevé Metric**
  - We can convert $\mathrm{d}r$ and $\mathrm{d}t$ in the Schwarzchild metric to $\mathrm{d}r_{\text{rain}}$ and $\mathrm{d}t_{\text{rain}}$ by first converting to the shell frame, and then using SR to convert to the rain frame. This gives [**only** for <u>free fall</u> from <u>rest</u> at infinity]:

$$\mathrm{d}t_{\text{rain}} = \mathrm{d}t + \beta_r\gamma_r^2\mathrm{d}r$$
$$\mathrm{d}r_{\text{rain}} = \mathrm{d}r$$

The resulting metric is

$$\boxed{\mathrm{d}\tau^2 = \left(1 - \beta_r^2\right)\mathrm{d}t_{\text{rain}}^2 - 2\beta_r \mathrm{d}t_{\text{rain}}\mathrm{d}r - \mathrm{d}r^2 - r^2\mathrm{d}\phi^2}$$

With $\beta_r \equiv \sqrt{2M / r}$. This provides a **global metric for the free fall frame**. It can be used *inside* and *outside* the horizon.

- We can use this to understand why light cannot escape back across the Horizon. Consider a beam of light falling or leaving radially ($\mathrm{d}\tau = 0$ and $\mathrm{d}\phi = 0$). The solutions to the above are:

$$\frac{\mathrm{d}r}{\mathrm{d}t_{\text{rain}}} = -\beta_r \pm 1$$

One corresponds to the inwards-moving photon, and the other to the outwards moving photon. Clearly, past $r = 2M$, even the outward going electron moves inwards in r coordinate.

- We can also use this expression to find the radial trajectory of light into our out of the black hole, by numerical integration.
- Applying variational calculus to the rain metric shows that constants of the motion for this fall are

$$\tilde{E}_{\text{rain}} = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t_{\text{rain}}}{\mathrm{d}\tau} - \sqrt{\frac{2M}{r}}\frac{\mathrm{d}r}{\mathrm{d}\tau} \qquad \tilde{L}_{\text{rain}} = \tilde{L}_{\text{bookkeeper}}$$

- **Radial *and* tangential motion**
  - In our case, we know that

$$\mathrm{d}\phi = \frac{L / m}{r^2}\mathrm{d}\tau$$

If we take the Schwarzchild metric and substitute the above for $\mathrm{d}\phi$ and use the expression for energy to substitute for $\mathrm{d}t$, and solve for $\mathrm{d}r$, we get

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

We multiply this by $\left(\mathrm{d}\tau\right)^2$ to get a formula for increments in ϕ and r as τ changes. This can, in principle, be used to solve for the entire orbit.

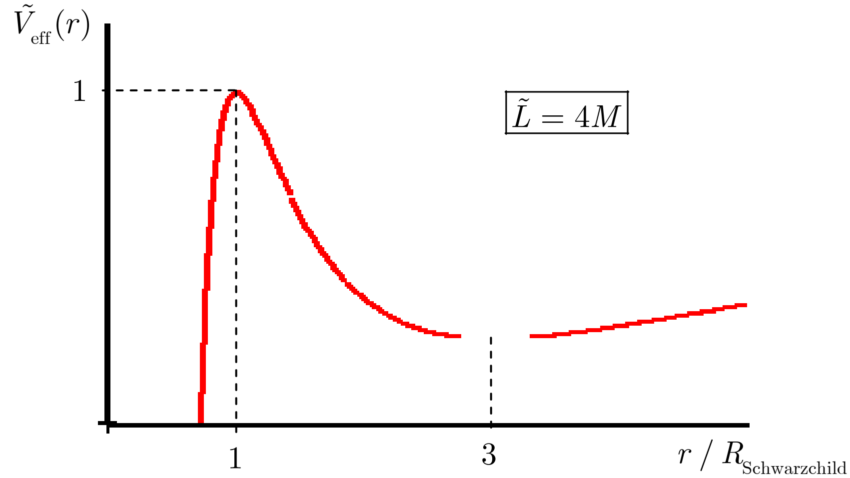
  - In Newtonian Mechanics, the effective potential is given by $\frac{1}{2}\dot{r} = \tilde{E} - \tilde{V}_{\text{eff}}$. By analogy, and looking at the equation above, we can define

$$\tilde{V}_{\text{eff}}^2(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

Such that

$$\left(\mathrm{d}r / \mathrm{d}\tau\right)^2 = \tilde{E}^2 - \tilde{V}^2$$

- The effective potential looks like



- A number of comments:
  - The "dip" is larger than in the Newtownian effective potential. This means that an elliptically orbiting satellite stays "closer" to the planet for longer. There is therefore precession of the orbit of the ellipse.
  - The case shown above is in fact the smallest possible stable circular orbit, with $r = 6M$.
  - Standard effective potential arguments help us analyse a general motion.

- **Light around a black hole**
  - For light, $\mathrm{d}\tau = 0$. We can use this, and the metric, to show that for the bookkeeper, the radial and tangential speed of light are not equal to 1.
  - For a shell observer, however, the speed of light is always 1 (we know this given that the metric for the shell observer is flat).
  - We can derive an "effective potential for light". Start with the metric

$$0 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 - r^2\left(1 - \frac{2M}{r}\right)\mathrm{d}\Omega^2$$

Then convert to shell coordinates, to get

$$\left(\frac{\mathrm{d}r_{\text{shell}}}{\mathrm{d}t_{\text{shell}}}\right)^2 = 1 - r^2\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}\Omega^2$$

We can write the RHS in terms of L and E

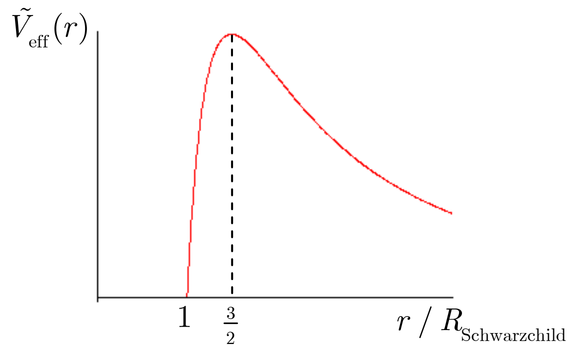
$$\left(\frac{\mathrm{d}r_{\text{shell}}}{\mathrm{d}t_{\text{shell}}}\right)^2 = 1 - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\frac{L^2}{E^2}$$

$$\frac{E^2}{L^2}\left(\frac{\mathrm{d}r_{\text{shell}}}{\mathrm{d}t_{\text{shell}}}\right)^2 = 1 - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)$$

And so by analogy, our effective potential here is

$$\boxed{V_{\text{eff}}^2(r) = \frac{1-(2M/r)}{r^2}}$$

This is totally independent of frequency and everything else. It is plotted below. Note, however, that this potential is expressed in terms of the **local** shell coordinates. It can therefore only be used for **qualitative estimations** of what will happen to the light.



There is therefore only one possible circular orbit, which occurs at r = $3M$.